MATHEMATICS

A TEXTBOOK FOR
CLASS XI

PART I

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Part I

IZHAR HUSAIN U. B. TEWARI
M.S. RANGACHARI D. D. JOSHI
V. KANNAN B. DEOKINANDAN

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Foreword

In keeping with the National Policy on Education (NPE) 1986, the National Council of Educational Research and Training (NCERT) developed a new curriculum in mathematics covering the entire school stage. It brought out new textbooks and other related instructional materials. The textbooks were developed by teams of authors consisting of eminent mathematicians with active interaction with user teachers. These textbooks have been in use in the schools affiliated to the Central Board of Secondary Education (CBSE) since 1988. Several states have also adopted/adapted these textbooks.

The Department of Education in Science and Mathematics (DESM) of the NCERT collected a lot of information from selected schools regarding the classroom use of the materials. In addition, very useful feedback was received by the DESM as well as by the authors themselves from several quarters. In the light of the feedback obtained, it was felt that the mathematics textbooks for Classes XI and XII needed some revision. Consequently, a small group consisting of Professor M. S. Rangachari, Professor A.M. Vaidya and Professor V. Kannan was set up to revise the books. Each of these authors separately consulted the teachers teaching Classes XI and XII through special workshops and finally the books were revised. I am grateful to all of them for consenting to take up the responsibility of revising the books and doing a good job within the short time available.

I also thank Prof. K. V. Rao, Dr B. Deokinandan, Shri Mahendra Shanker, Dr S. K. S. Gautam, Dr Ram Avtar, Dr Hukum Singh and Shri G. D. Dhall of the DESM who, besides contributing to material development took lots of pains to see the books through the press. I am indebted to the teachers, students, parents and institutions who favoured us with valuable comments and suggestions which formed the basis for the revision.

Though some improvements have been made in the present revised books, there will always be ample scope for further improvement. So, suggestions/comments from users and others are most welcome.

A. K. Sharma
Director
National Council of Educational
Research and Training

SCIENCE RELATED VALUES

Curiosity, quest for knowledge, objectivity, honesty and truthfulness, courage to question, systematic reasoning, acceptance after proof/verification, open-mindedness, search for perfection and team spirit are some of the basic values related to science. The processes of science, which help in searching the truth about nature and its phenomena are characterised by these values. Science aims at explaining things and events. Therefore to learn and practise science :

* Be inquisitive about things and events around you.
* Have the courage to question beliefs and practices.
* Ask 'what', 'how' and 'why' and find your answers by critically observing, experimenting, consulting, discussing and reasoning.
* Record honestly your observations and experimental results in your laboratory or outside it.
* Repeat experiments carefully and systematically if required, but do not manipulate your results under any circumstance.
* Be guided by facts, reasons and logic. Do not be biased in one way or the other.
* Aspire to make new discoveries and inventions by sustained and dedicated work.

Preface

The textbooks of mathematics for senior secondary classes developed during 1987-88 by the author-team constituted by the NCERT have been generally appreciated by their users in schools as well as by others interested in mathematics education. However, in course of time, some good suggestions have been received by the authors. Also, the Department of Education in Science and Mathematics, NCERT gathered some feedback from various schools in which the textbooks have been taught.

The NCERT desired that these textbooks be revised in the light of the feedback received from several quarters and entrusted the job of revision to Prof. A.M. Vaidya (as a non-author), Prof. Kannan and myself (as among the authors of these textbooks). We three went through the textbooks critically. Further, all the three of us conducted small workshops separately inviting teachers teaching these materials in their schools and other experienced in teaching at the senior secondary and higher levels to discuss the revision. After revising the material in the workshops they were further more closely scrutinised with the help of Dr V. K. Krishnan and Kum. R. Vijaylakshmi before they were passed on to the NCERT for printing.

The main features of the revision are:

(i) Change of sequence of topics/concepts in the textbook of Class XI to ensure more logical development and inclusion of historical references at the appropriate places in the text itself rather than as an appendix.

(ii) Inclusion of some additional concepts like moment of a couple, Bayes' formula, etc. either introduced by CBSE and other boards or to plug loopholes in the earlier version of the text. (The concept of rank of a matrix is introduced to treat more precisely the consistency of a system of equations.)

(iii) Inclusion of additional exercises.

(iv) Elimination of errors that had inadvertently crept into the earlier version of the textbooks.

I am very grateful to Prof. A. M. Vaidya and Prof. Kannan for the pains they have taken to complete the work in a short time in spite of their various other commitments. I must make special mention of the valuable contribution made by Dr V. K. Krishnan and Kum. R. Vijaylakshmi who went through the minute details in the revised manuscript and helped in the further refinement of the manuscript. I must thank Prof. K. V. Rao, Dr B. Deokinandan and Shri Mahendra Shanker of the Department of Education in Science and Mathematics, NCERT who initiated the revision, gave all support to us and finally saw the revised books through the press. Lastly, I am very grateful to the teachers who attended the workshops for revision of the textbooks and many other friends who made valuable comments/suggestions that helped in the revision of the

textbooks. The production team of the Publication Department of the NCERT deserve all appreciation for the good quality of production and expeditious printing. In spite of all the care taken so far it is possible that there are still some shortcomings in the revised textbook. I shall be grateful for any suggestions or comments which will help the further improvement of the material.

M.S.Rangachari
The Ramanujan Institute
University of Madras

Contents

CHAPTER 1

Sets, Relations and Functions

1.1 Set

This chapter is intended mainly as a review of the topics discussed in the textbooks of lower classes. You have now become familiar with the concept of 'set'. As you know a set goes along with the concepts: "objects", "belongs to" or "does not belong to" the set. If S is a set and a belongs to the set S, we write "$a \in S$", which is read as 'a belongs to S' or 'a is an element of S' or 'a is in S'. In Fig. 1.1 if S is a set of points of which a is one, then $a \in S$.

If a does not belong to S, we write $a \notin S$, a is then not an element of S. In mathematics, we are mostly concerned with sets of numbers, or sets of other mathematical entities e.g. sets of polynomials, sets of fractions, sets of lines, sets of circles and so on. A rigorous theory of sets has been developed on the basis of axioms but for our purpose an intuitive approach known as "naive set theory"* would be enough. A set is determined, if, given any object, it can be said whether the object belongs to or does not belong to the set.

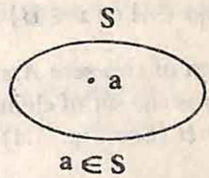

Fig.1.1

In any context, we have in mind some set and we consider different subsets of this set. This set is called the *universal set*. For example, when in two dimensional geometry we discuss sets of lines or triangles or circles, then the universal set may be the plane in which the lines, circles and triangles lie. Suppose we are discussing integers, positive integers, or prime numbers, then we can take the universal set to be $\mathbb{Z}$, the set of integers. We can also take $\mathbb{R}$, the set of real numbers, as the universal set in this case. Thus the universal set is determined by the context of the problem discussed. You also know what is meant by the empty set or the void set, or the null set which is denoted by the symbol ϕ, which is the letter 'oh' of the Scandinavian alphabet.

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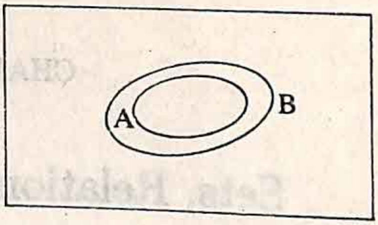
* A nicely written book with this title by P.R. Halmaos is recommended for further study.

If A and B are two sets such that every element of A is also an element of B, then we say that 'A is a subset of B'. In symbols, we write $A \subset B$ (See Fig. 1.2). For example, $\mathbf{Z} \subset \mathbf{Q}$, (where $\mathbf{Q}$ denotes the set of rational numbers), since every integer is a rational number. It may be noted that $A \subset A$ and the empty set is a subset of A for every set A. If $A \subset B$, we sometimes write $B \supset A$ and read 'B contains A' or 'B is a superset of A'. We can also say 'A is contained in B'.

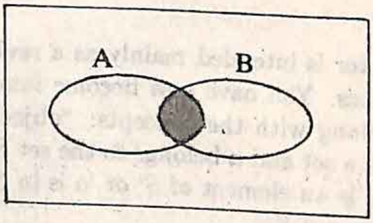


$A \subset B$

Fig 1.2

Two sets are said to be equal if they have the same elements, more precisely, elements of A are in B and *vice versa*. You can see that if $A \subset B$ and $B \subset A$, then $A = B$.

Let us recall that the union of two sets A and B is the set consisting of all the elements of A together with all the elements of B. There is no point here in repeating the elements more than once. The union of two sets A and B is written as $A \cup B$ (See Fig. 1.3).



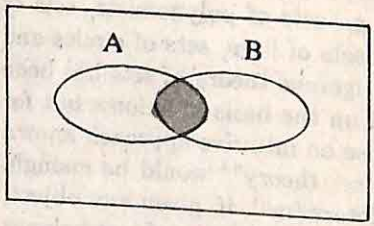
$A \cup B$

Fig 1.3

In symbols, we have

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

The intersection of two sets A and B, denoted by $A \cap B$ is the set of elements common to A and B (See Fig. 1.4). In symbols, we have



$A \cap B$

Fig 1.4

$$A \cap B = \{x | x \in A \text{ and also } x \in B\}$$

For example, if A is the set of positive integral multiples of 2 and B is the set of positive integral multiples of 3,

$$\text{i. e., } A = \{x | x = 2n, n = 1, 2, 3, \ldots\} = \{2, 4, 6, 8, \ldots\}$$
$$B = \{x | x = 3n, n = 1, 2, 3, \ldots\} = \{3, 6, 9, \ldots\}$$

then

$$\begin{aligned} A \cup B &= \{x | x \text{ is a multiple of 2 } \textit{or} \text{ a multiple of } 3\} \\ &= \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, \ldots\} \end{aligned}$$

and $A \cap B = \{x | x \text{ is a multiple of both 2 and 3}\}$
$= \{6, 12, 18, \ldots\}$
$= \{\text{all the multiples of } 6\}$

We can similarly define the union and intersection of any number of sets.

If U is the universal set and $A \subset U$, then the complement of A with respect to U denoted by A^c, is the set of all those elements of U which do not belong to A (See Fig. 1.5).

In symbols, we have

$$A^c = \{x | x \in U, x \notin A\}$$

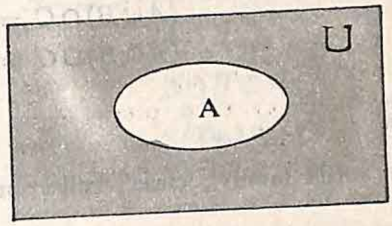


Fig 1.5

Evidently

$$(A^c)^c = A$$
$$U^c = \phi$$

De Morgan's Laws

If A and B are two subsets of U, then it can be shown that

$$(A \cup B)^c = A^c \cap B^c$$

and

$$(A \cap B)^c = A^c \cup B^c$$

These relations are true even for more than two sets. These results may be verbally stated as follows:

Complement of union is equal to intersection of complements and complement of intersection is equal to union of complements.

We use the notation $\bigcup_{i=1}^{n} A_i$ to denote the union of n sets $A_1, A_2, A_3, \ldots, A_n$. Thus,

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \ldots \cup A_n$$

Similarly, we use the notation $\bigcap_{i=1}^{n} A_i$ to denote the intersection of n sets $A_1, A_2, \ldots, A_n$.

Thus,

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \ldots \cap A_n$$

The difference of two sets A and B, denoted by $A - B$, is defined by

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$

$A - B$ is also denoted by $A \setminus B$.

It may be noted that unions and intersections are commutative, associative and each of them is distributive over the other, i.e.

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Remarks

(i) The words 'family', 'class', 'collection' are also used as synonyms for the word 'set'

(ii) If $A \cap B = \phi$, we say A and B are disjoint. If $A_1, A_2 \ldots$ is a sequence of sets, it is said to be a pairwise disjoint family of sets if and only if any two sets of this family are disjoint. For example, if A is the set of all odd integers and B, that of even integers, then A and B are disjoint. The class of sets $\{A_2, A_3, A_5, A_7\}$, where A_2, A_3, A_5, A_7 are defined by

$$\begin{aligned} A_2 &= \{2, 2^2, 2^3, 2^4, 2^5, \ldots\} \\ A_3 &= \{3, 3^2, 3^3, 3^4, 3^5, \ldots\} \\ A_5 &= \{5, 5^2, 5^3, 5^4, 5^5, \ldots\} \\ \text{and } A_7 &= \{7, 7^2, 7^3, 7^4, 7^5, \ldots\} \end{aligned}$$

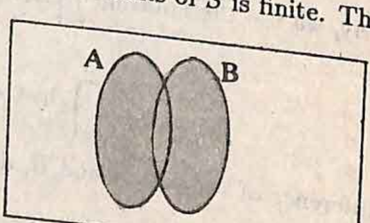
is pairwise disjoint.

Note that ϕ is such that $\phi \subset A$ for any A and $\phi \cap A = \phi$ for any A. i.e; the null set is contained in every set and at the same time disjoint from every set.

(iii) A set S is said to be a finite set if the number of elements of S is finite. The null set ϕ is regarded as a finite set

Example 1.1

Show that (i) $A \subset A \cup B$
(ii) $A \cap B \subset A$



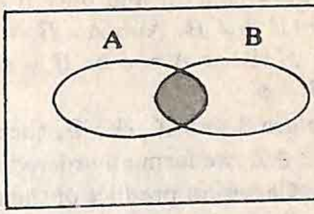
$A \subset A \cup B$

Fig 1.6

Solution

(i) Let $x \in A$. Then $x \in A$ certainly, while x may belong to B or may not belong to B. So in either case, $x \in A \cup B$. Hence, $A \subset A \cup B$. (See Fig.1.6).

(ii) If $x \in A \cap B$, then $x \in A$ and $x \in B$. So in particular, $x \in A$. Hence $A \cap B \subset A$. Similarly $A \cap B \subset B$ (See Fig. 1.7).



$A \cap B \subset A$

$A \cap B \subset B$

Fig.1.7

Remark

You might have used Venn diagrams in lower classes to verify set theoretic facts. It is, however, necessary to draw the diagrams in the most general way. For example, suppose we have to represent three sets. Then Fig. 1.8 does not represent the general case, since it would follow that $A \cap B \cap C = \phi$, which need not always be the case. Fig. 1.9 represents the general case.

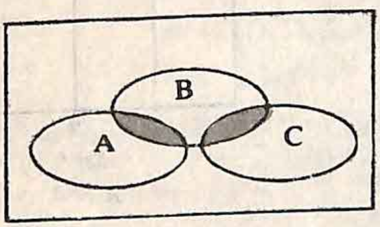


Fig.1.8

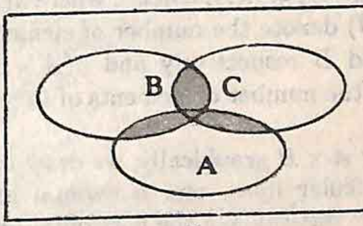


Fig.1.9

1.2 Cartesian Product of Sets, Relations

The procedure of considering union or intersection of a class of sets and the difference between two given sets is to create more sets out of given sets. Yet another procedure is to consider what is known as the Cartesian product of two or more sets. For simplicity, let A, B be two sets. By an ordered pair of elements we mean a pair (a, b), $a \in A$, $b \in B$ in that order. (a, b), (b, a) are different unless $a = b$ i.e. a and b are one and the same object. The set of all ordered pairs (a, b), of elements $a \in A$, $b \in B$ is called the *Cartesian product* of the sets A and B and is denoted by $A \times B$. In symbols,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

We say $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. Clearly, $A \times B$ and $B \times A$ are different sets if $A \neq B$. Also $A \times B = \phi$ when one or both of A, B are empty. Conversely if $A \times B = \phi$, either $A = \phi$ or $B = \phi$. Again, $A \times B = B \times A$ if and only if $A = B$ or $A = \phi$ or $B = \phi$.

If there are 3 sets X, Y, Z, then choosing 3 elements x,y and z such that $x \in \mathbf{X}$, $y \in \mathbf{Y}$ and $z \in \mathbf{Z}$, we form an ordered triplet (x, y, z). The set of all such ordered triplets is called the Cartesian product of the three sets X, Y, Z and is denoted by $X \times Y \times Z$. We can, similarly, form the Cartesian product of n sets. Clearly, each element of the Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is an ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$ and so on. The Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \times A_2 \times \ldots \times A_n$ or briefly by $\prod_{i=1}^{n} A_i$.

Example 1.2

Let $A = \{1, 2, 3\}$ and
$B = \{2, 4\}$

Find $A \times B$ and show it graphically.

Solution

Clearly $A \times B = \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4)\}$

It may be guessed from the above example that $n(A \times B) = n(A) \times n(B)$, where $n(A)$ and $n(B)$ denote the number of elements of A and B respectively and $n(A \times B)$ denotes the number of elements of the set $A \times B$.

To show $A \times B$ graphically, we draw two perpendicular lines, one horizontal and the other vertical. On the horizontal line, we represent the elements of A and on the vertical line, the elements of B(Fig. 1.10).

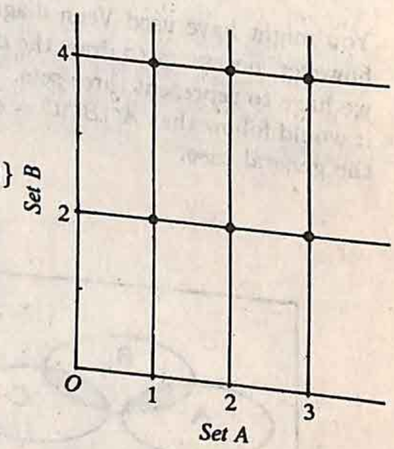


Fig.1.10

If $a \in A$, $b \in B$, we draw a vertical line through a and a horizontal line through b. They will meet in a point which will denote the ordered pair (a, b). The set of points so obtained graphically represents $A \times B$.

In particular, if A is the set of all real numbers, we can deem A to consist of all points in a line. $A \times A$ will then consist of all points in the plane. If P is a point in the plane, then a and b in the corresponding ordered pair (a, b) are called the coordinates of P.

We now consider a set B of persons, as follows:

$$B = \{\text{Asha,Zarina, Mary, Sushma}\}$$

Let us consider another set A of their brothers as follows:

$$A = \{\text{Ram,Kedar,Jamil,Albert}\}$$

There is a relation 'is a brother of' between the elements of the sets A and B. If we write R for the relation "is a brother of" and if Asha has two brothers Ram and Kedar, Zarina has a brother Jamil and Mary has a brother Albert, then the above information can be represented as:

Ram R Asha, Kedar R Asha, Jamil R Zarina, Albert R Mary.

Omitting the letter R between the pairs of names and writing the pair of names as an ordered pair, the above information can also be written as a set of ordered pairs R where

$$\begin{aligned} R &= \{(\text{Ram,Asha}),(\text{Kedar,Asha}),(\text{Jamil,Zarina}), (\text{Albert,Mary})\} \\ &= \{(x, y)|x \in A, y \in B, xRy\} \end{aligned}$$

Thus we see that the relation "is a brother of " from set A to set B gives rise to a subset R of $A \times B$ such that $(x, y) \in R$ if and only if xRy. Let us consider another example. Let $\mathbf{N}$ be the set of natural numbers. Consider the relation 'has as its square' from the set $\mathbf{N}$ to $\mathbf{N}$. If we write R for 'has as its square', then we get the statements:

$$1R1, 2R4, 3R9, 4R16 \ldots$$

Again, we omit R between the pairs of numbers and write them as ordered pairs. We thus find that the relation 'has as its square' gives rise to a set R of ordered pairs where

$$\begin{aligned} R &= \{(1,1), (2,4), (3,9), (4,16), (5,25) \ldots\} \\ &= \{(x, y)|x, y \in \mathbf{N} \text{ and } y = x^2\} \end{aligned}$$

The set $\mathbf{R}$ obtained from the relation 'has as its square' from $\mathbf{N}$ to $\mathbf{N}$ is subset of $\mathbf{N} \times \mathbf{N}$. Again we note that $(x, y) \in R$ if and only if $y = x^2$.

Keeping the above examples in the background, we can now define a relation.

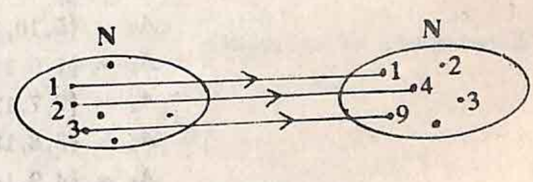


Fig.1.11

Definition

A relation R from a set A to a set B is a subset of $A \times B$. A is called the *domain* of R and B is called the *co-domain* of R.

The set of second entries of the ordered pairs in a relation is called the *range* of the relation.

Then in the example in Fig. 1.11

$$\text{domain} = \{1, 2, 3, \ldots\}$$

$$\text{and range} = \{1, 4, 9, \ldots\}$$

If $(a, b) \in R$, then we write it as aRb and read it as 'a is in relation R to b'. If $A = B$, then the relation is called a relation defined in A or simply a relation in A.

A relation R in A is said to be *reflexive* if aRa for all $a \in A$. It is said to be *symmetric* if aRb implies bRa. It is said to be *transitive* if aRb and bRc together imply aRc. Let a, b be two triangles in a class of all triangles in a plane and let aRb be "a is congruent to b". Then we know that R is a relation which has all the properties mentioned above. Any such relation which is reflexive, symmetric and transitive is called an *equivalence relation.*

Not all relations are equivalence relations. In the set N of natural numbers aRb $a, b \in N$ defined by 'a divides b' or in symbols, $a|b$, is a reflexive relation but not symmetric. It is, however, transitive. So it is not an equivalence relation.

An important property of an equivalence relation is that it divides the set into pairwise disjoint subsets whose collection is called a *partition* of the set. Note that the union of the subsets in the collection is the whole set. We illustrate this point by the following example.

In the set N of natural numbers, we define a relation R as follows:

For $n, m \in \mathrm{N}$, nRm if on division by 5 each of the integers n and m leaves the same remainder i.e. one of the numbers 0,1,2,3 and 4. It is easily seen that R is an equivalence relation. For, aRa for all $a \in \mathrm{N}$ (Reflexive). If aRb, then bRa for $a, b \in \mathrm{N}$ (Symmetric). If aRb, and bRc, then aRc (Transitive).

Let $A_o = \{\, n | n \in \mathrm{N}$ and on division by 5, n leaves the remainder 0 $\}$

$A_1 = \{\, n | n \in \mathrm{N}$ and on division by 5, n leaves the remainder 1 $\}$

Similarly, we define sets A_2,A_3, and A_4. Since there can be only five remainders viz. 0,1,2,3,4 on division by 5, we shall not get any other sets.

Now,

$$\begin{aligned} A_0 &= \{5, 10, 15, 20, \ldots\} \\ A_1 &= \{1, 6, 11, 16, 21, \ldots\} \\ A_2 &= \{2, 7, 12, 17, 22, \ldots\} \\ A_3 &= \{3, 8, 13, 18, 23, \ldots\} \\ A_4 &= \{4, 9, 14, 19, 24, \ldots\} \end{aligned}$$

It is evident that the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{1=0}^{4} A_i = \mathrm{N}$$

We have thus seen that the equivalence relation R, defined in this example, has divided the set N into five pairwise disjoint subsets.

Conversely, a partition of a set defines an equivalence relation. If $S_1, S_2, \ldots S_n$ is a partition, this equivalence relation is aRb if and only if $a, b \in S_i$ for some $i = 1, 2, \ldots, n$.

Example 1.3

If $A = \{1,2\}$, $B = \{3,4\}$, $C = \{4,5\}$, what is $A \times (B \cup C)$?

Solution

$B \cup C = \{3,4,5\}$ So, $A \times (B \cup C) = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\}$

Example 1.4

Prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Solution

If $(x, y) \in A \times (B \cap C)$, then $x \in A$ and $y \in B \cap C$.
So, $x \in A, y \in B$ and $x \in A, y \in C$ i.e. $(x,y) \in A \times B$ and $(x,y) \in A \times C$
Hence, $(x,y) \in (A \times B) \cap (A \times C)$
Hence, $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$ (i)
Conversely, if $(x,y) \in (A \times B) \cap (A \times C)$, then $(x,y) \in A \times B$ and $(x,y) \in A \times C$.
So, $x \in A$, and $y \in B$ and $y \in C$ i.e. $x \in A$ and $y \in B \cap C$.
Hence, $(x,y) \in A \times (B \cap C)$
Hence, $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$ (ii)
From (i) and (ii), we get

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Example 1.5

If $a, b \in N$ and R is "a is a divisor of b", then R is a relation on N. The subset S o $N \times N$ which corresponds to the relation is

$$S = \{(n, rn) : n \in N, r \in N\}$$

For instance (1,3),(3,15), (4,4) are in S while (2,5),(3,7) do not belong to S.

Example 1.6

If $a, b \in \mathbf{R}$, the set of all real numbers and R is "$\mid a - b \mid$ is a rational number", then R is a relation on $\mathbf{R}$. The subset of $\mathbf{R} \times \mathbf{R}$ which define the relation is

$$S = \{(a, \underline{a} + a) : a \in R, \underline{a} \in Q, \text{ the set of all rational numbers }\}$$

Here $(1, 2\frac{1}{2}), (-\sqrt{2}, \frac{3}{2} - \sqrt{2}), (\pi, \pi - \frac{1}{2})$
are in S. $(\sqrt{2}, \pi + \sqrt{2}) \notin S$.

Example 1.7

If $a, b \in \mathrm{N}$ and aRb if "$b - a$ is divisible by a fixed number $m \in \mathrm{N}$" then R is a relation on N. The subset S of $N \times N$ corresponding to the relation is

$$S = \{(n, n + rm) : n \in N, r \in \mathrm{N}\}$$

If $m = 3, (2,8), (5,11) \in S$ while $(3,8) \notin S$.

Note: A relation S which is a subset of $A \times B$ may not be such that the ordered pairs in S exhaust A in their first entries.

1.3 Functions

The concept of a function is a special case of that of a relation. To be specific, while a relation may relate an element of the domain to more than one element of the range, a function relates each element of the domain to one and only one element of another set viz. the co-domain. In other words, a function is a single-valued association of all the elements of the domain with elements of the co-domain. Thus, if $\mathbf{R}^+$ denotes the set of all non-negative real numbers, f, defined by

$$f(x) = \text{ square root of } x, x \in \mathbf{R}^+,$$

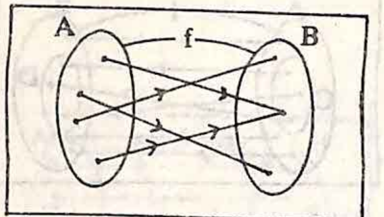
defines only a relation while the definition

$$f(x) = \text{ non-negative square root of } x, x \in \mathbf{R}^+,$$

gives a function.

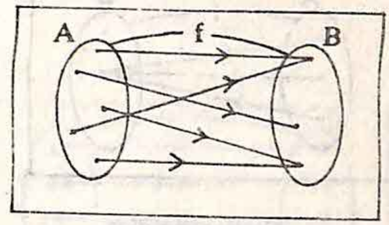
If $f : A \to B$ is a function, by the *graph of* f is meant the subset $\{(a, f(a)) | a \in A\}$ of $A \times B$. Two functions $f, g : A \to B$ are equal (or the same) if and only if for each $a \in A, f(a) = g(a)$. This is equivalent to saying that the graph of f and the graph of g are one and the same set. We need not, therefore, distinguish a function from its graph. If $f : \mathbf{R} \to \mathbf{R}$, we note that the graph of f is precisely the graph in the usual sense with reference to a pair of rectangular axes. Hence the terminology. Incidentally, the graph of a function $f : A \to B$ is precisely the subset of $A \times B$ which is determined by the relation which is given by the function f.

Let $f : A \to B$ be a function. Then $A = dom f$ and $B = codom\, f$. f is said to be a function defined on A (or map of A) *into* B(See Fig. 1.12).



Into function

Fig 1.12



Onto function (surjection)

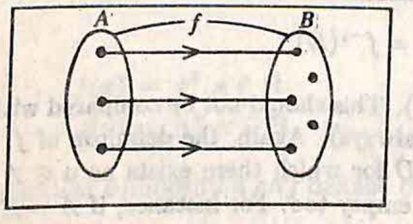
Fig.1.13

If the range of f i.e. $ran f$, is such that $ran f = B$, f is said to be a function defined on A (or map of A) *onto* B (See Fig. 1.13). Note that $f : A \to B$, which is *onto*, is also into B.

Such a function is also sometimes said to be *surjective*. If distinct elements of A are taken to distinct elements of B by f, i.e.

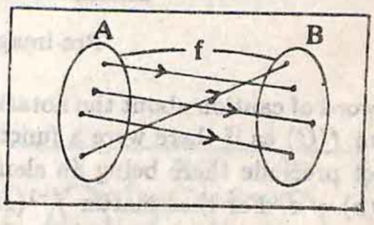
If $x_1, x_2 \in A, x_1 \neq x_2$ implies $f(x_1) \neq f(x_2), f$ is said to be *one-to-one* or, sometimes, *injective* function or map (See Fig. 1.14).

A map (or function) which is both one-to-one (injective) and onto (surjective) is said to be a *bijective* map (or function) or simply a *bijection* (See Fig. 1.15).



One-one into function (injective)

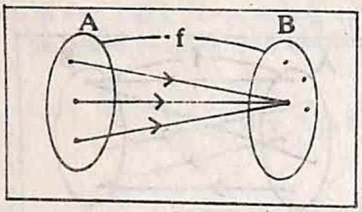
Fig 1.14



One-one onto function (bijection)

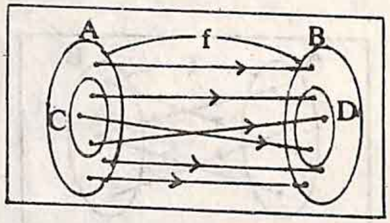
Fig.1.15

If $f(a) = b$ for every $a \in A$ and for a fixed $b \in B$, f is said to be *constant* map (See Fig. 1.16). A constant map cannot obviously be one-to-one, if its domain has more than one element. If $f : A \to B$ is a map and $C \subset A$, then we write $D = f(C) = \{b | b \in B, b = f(c) \text{ for some } c \in C\}$ and call $f(C)$ the *image* of C by f (See Fig. 1.17).



Constant function

Fig.1.16



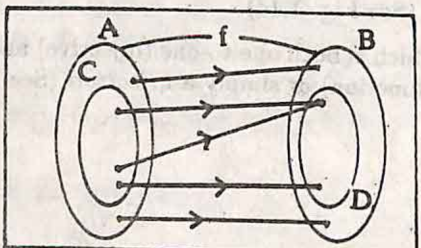
$D = f(C)$, image of C by f

Fig.1.17

In this notation $f(A) = ran f$. f is a map of A onto B if and only if $f(A) = B$. If $D \subset B$, then we write

$$f^{-1}(D) = \{a | a \in A, f(a) = d \text{ for some } d \in D\}$$

and call $f^{-1}(D)$ the *pre-image, total original or pull back* of D (See Fig. 1.18).



Pre-image of D $C = f^{-1}(D)$

Fig.1.18

A word of caution about the notation $f^{-1}(D)$. This should not be compared with the notation $f(C)$ as if there were a function f^{-1} (always). Again, the definiton of $f^{-1}(D)$ does not preclude there being an element $d \in D$ for which there exists no $a \in A$ such that $f(a) = d$. For that matter $f^{-1}(D)$ can be empty too. For instance, if $A = B = \mathbf{R}$ and $f : A \to B$ is defined by $f(a) = [a]$, the largest integer less than or equal to $a \in A$, and $D = \{\frac{1}{2}, \frac{3}{4}\} \in \mathbf{R}$, then $f^{-1}(D) = \phi$.

Let us now consider a bijection $f : A \to B$. It is then clear that for any $b \in B$ there exists one and only one $a \in A$ such that $f(a) = b$. Define the association f^{-1} of elements of B with elements of A as follows:

$$f^{-1}(b) = a \text{ if and only if } f(a) = b.$$

The observation made just above shows that this association is a function or map; viz. $f^{-1} : B \to A$. It is easy to verify that f^{-1} is a bijection too. The function f^{-1} is called the inverse of the function f (See Fig. 1.19 (a) and (b)).

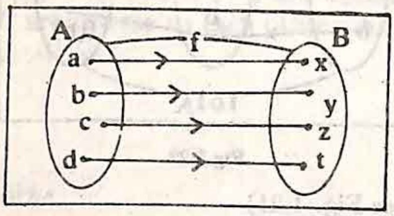


Fig.1.19 (a)

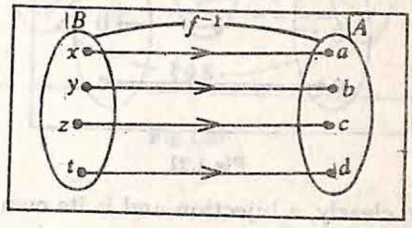


Fig.1.19 (b)

To sum up, any bijection of a set onto another has *inverse* which is also a bijection.

It is not difficult to see that $(f^{-1})^{-1} = f$. Let $f : A \to B$, $g : B \to C$, be two functions. Then the function $g \circ f$ defined by

$$(g \circ f)(a) = g\,(f(a))\,, a \in A.$$

$g \circ f : A \to C$ is called the *composition* of f and g (See Fig. 1.20).

For example, if $A = B = \mathbf{R}$ and $C = \mathbf{Z}$, the set of all integers, f, g defined respectively by

$$f(x) = x^2, x \in \mathbf{R},$$
$$g(y) = [y]\,, y \in \mathbf{R}.$$

have for their composition $g \circ f$ defined by

$$g \circ f(x) = \left[x^2\right], x \in \mathbf{R}.$$

The *identity map of a set A* into itself, denoted by I_A, is defined by

$$I_A(a) = a, a \in A.$$

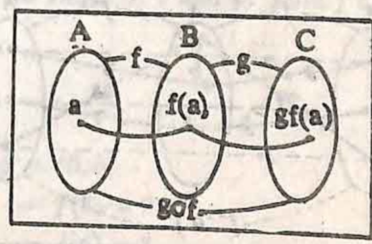


Fig.1.20

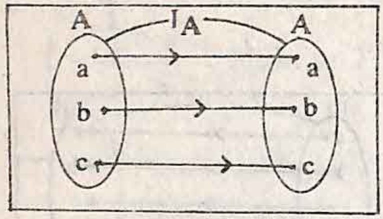


Fig.1.21

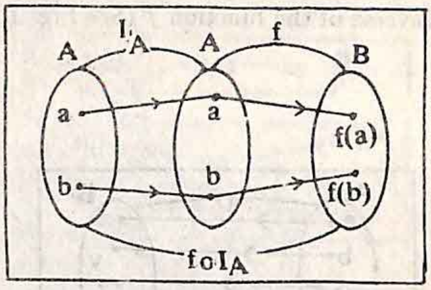


Fig.1.22

I_A is, clearly, a bijection and is its own inverse (See Fig. 1.21).
$f : A \to B$ is map, then it follows that $f \circ I_A = f$ (See Fig. 1.22).
Similarly, $I_B \circ f = f$.
In particular, if $f : A \to A$ is a map (called by some authors a self map), then

$$f \circ I_A = I_A \circ f = f.$$

Moreover, for any two maps $f, g : A \to A$, $g \circ f$ is defined and $g \circ f : A \to A$. If now, $f : A \to B$ is a bijection so that it has inverse f^{-1}, then

$$f^{-1} o f = I_A \text{ and } f \circ f^{-1} = I_B$$

If, in particular, $A = B$, so that f is a bijection of A onto itself, then

$$f^{-1} o f = f \circ f^{-1} = I_A$$

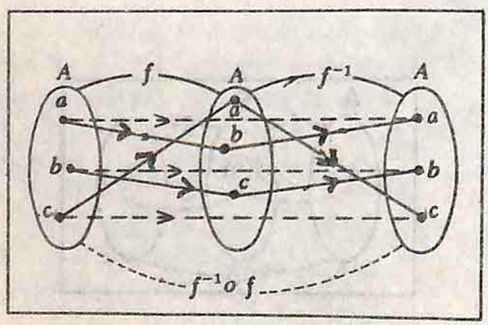


Fig.1.23

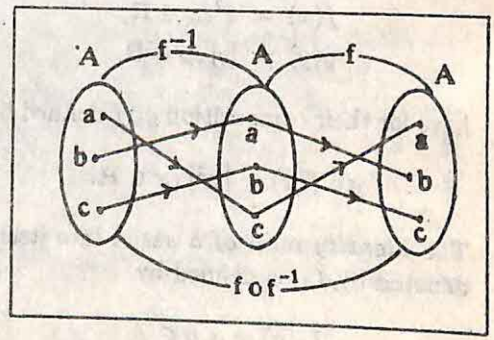


Fig.1.24

(See Fig. 1.23 and 1.24). Moreover, if $f : A \to A$, $g : A \to A$ are mappings which are such that

$$f \circ g = g \circ f = I_A$$

(See Fig. 1.25) then f, g are bijections which are inverse to each other. Justify this assertion.

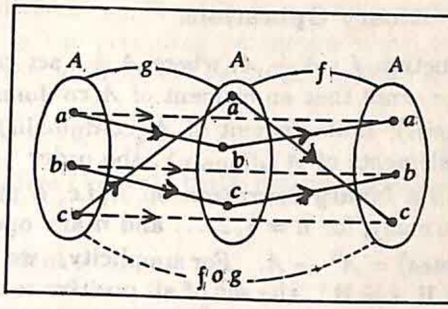


Fig.1.25

Remarks

At this stage we can rigorously define a set to be *finite* if there exists a bijection of the set on to the set $N_n = \{1, 2, 3, \ldots, n\}$ for some natural number n. The *void* set ϕ is taken to be finite.

Example 1.8

If $A = \{1, 2, 3\}$ and f, g are relations corresponding to the subsets of $A \times A$ indicated against them, which of f, g is a function? Why?

$$f = \{(1,3), (2,3), (3,2)\}$$
$$g = \{(1,2), (1,3), (3,1)\}$$

Solution

f is a function since each element of A in the first place in the ordered pairs goes with only one element of A in the second place. g is not a function because 1 is related to both 2 and 3.

Example 1.9

If $f = \{(5,2), (6,3)\}$, $g = \{(2,5), (3,6)\}$, what is the range of f and g? Find $f \circ g$.

Solution

$$ran\ f = \{2,3\}, ran\ g = \{5,6\}$$
$$f \circ g(2) = f(g(2)) = f(5) = 2$$
$$f \circ g(3) = f(g(3)) = f(6) = 3$$
$$\text{Hence, } f \circ g = \{(2,2), (3,3)\}$$

1.4 Binary Operations

A function $f : A \to A$, where A is a set can also be thought of as a *unitary operation* in the sense that an element of A(co-domain) is associated to each singleton subset of A (domain). If an element of A (co-domain) is associated uniquely with every subset of two elements of A (domain), the order of the elements being taken into account, we obtain a *binary operation* on A, i.e. a map $A \times A \to A$ is called a binary operation in A. Formally, for $n = 1, 2, \ldots$ and n-ary operation on the set A is a map $f : A \times A \times \ldots \times A$ (n *times*) $= A^n \to A$. For simplicity, we consider here unitary and binary operations only. If $A = \mathbf{R}^+$, the set of all positive real numbers, taking reciprocals, or, what is the same, the map $x \to \frac{1}{x} : A \to A$ is a unitary operation.

If $A = \mathbf{R}$, the set of all real numbers, $(x, y) = x + y : \mathbf{R}^2 \to \mathbf{R}$, or, what is the same, addition of two real numbers, is a binary operation on $\mathbf{R}$. Multiplication is also a binary operation on $\mathbf{R}$. However, division is not a binary operation on $\mathbf{R}$, since division by 0 is not defined. But division is a binary operation on $\mathbf{R} \setminus \{0\}$. We can think of other binary operations in terms of these known binary operations. For instance, $(m, n) \to m + n + mn : \mathbf{Z}^2 \to \mathbf{Z}$, $\mathbf{Z}$ being the set of all integers, is a binary operation on $\mathbf{Z}$. Keeping the map involved in the background, we prefer to speak of the binary operation '+' or addition instead of the map $(x, y) \to x + y$ or of the binary operation '$\cdot$', or multiplication instead of the map $(x, y) \to xy$. With an analogous notation we can speak of the binary operation '$\circ$' by writing $mon = m + n + mn$ on $\mathbf{Z}$.

We pointed out that when defining a binary operation the order of the elements is to be taken into account; in other words, the map which defines the binary operation on A is on the set A^2 of all *ordered* pairs of elements of A. If $\circ$ is the operation, for $a, b \in A$, aob and boa may be different elements of A. If, however, for every pair a, b of elements A.

$$a \circ b = b \circ a,$$

then the binary operation is *commutative*. For example, the binary operations $+, \cdot$, on $\mathbf{R}$ are commutative. Is the operation $\circ$ defined at the end of the last paragraph commutative? The operation of division in $\mathbf{R} - \{0\}$ is evidently *not* commutative. If the binary operation $*$ on $\mathbf{Z}$ is defined by

$$m * n = m - n + mn$$

then $1 * 2 = 1 - 2 + 1 \cdot 2 = 1$ while $2 * 1 = 2 - 1 + 2 \cdot 1 = 3$ so that $1 * 2 \neq 2 * 1$ and $*$ is *not* a commutative binary operation.

If $\circ$ is a binary operation on a set A and a, b, c are three elements of A, with due regard to the order in which a,b,c occur, we can consider the two elements

$$ao(boc), (aob)oc.$$

There is no *prima facie* reason for these to be one and the same. If, however, they are one and the same for *every* ordered triplet a, b, c of elements of A, then $\circ$ is said to be an *associative* operation. As examples of associative binary operations, we have addition

and multiplication defined on $\mathbf{R}$. Clearly, division on $\mathbf{R}-\{0\}$ is not an associative operation (Why?). The operation $*$ defined in the preceding paragraph which is not commutative is not also associative. If X is the set$\{a, b\}$ let the binary operation $\circ$ be defined by

$$a \circ a = a, b \circ b = b, a \circ b = b, b \circ a = a. \tag{1.1}$$

This operation is associative but *not* commutative (Verify). If the binary operation $\circ$ is defined on Z by

$$\ell \circ m = \frac{\ell + m}{2}, \ell, m \in Z,$$

the operation $\circ$ is evidently commutative. It is easily verified that this operation is *not*, however, associative. To sum up, the concepts of associativity and commutativity of a binary operation are independent.

If $\circ$ is a binary operation on X and if there is $e \in X$ such that $aoe = eoa = a$, e is said to be an *identity element* for the operation. For instance, for the binary operation of addition in $\mathbf{R}$, 0 is the identity element. For multiplication it is 1.

Remark

It is sometimes convenient to write down a binary operation by means of a table. For instance, the operation defined by (1.1) above is written in the form :

o	a	b
a	a	b
b	a	b

The presence of one or more binary operations in a set gives a structure for the set which could be studied. For instance, we have because of the availability of such operations, the concepts of group, ring, vector space, field, etc. The study of these structures is generally known as algebra.

Example 1.10

In the set $\mathbf{N}$ of natural numbers, define the binary operation $\circ$ by

$$m \circ n = \text{g.c.d}\ (m, n), m, n \in \mathbf{N}.$$

Is the operation commutative, associative?

Solution

The operation is clearly commutative since

$$\text{g.c.d}\ (m, n) = \text{g.c.d}\ (n, m)$$

It is also associative because for $\ell, m, n \in \mathbf{N}$,

$$\text{g.c.d}\ (\ell, \text{g.c.d}\ (m, n)) = \text{g.c.d}\ ((\text{g.c.d}\ (\ell, m), n))$$

Example 1.11

Let N^k be the set of all ordered k-tuples of natural numbers. If $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$, $x_i, y_i \in N, i = 1, 2, \ldots, k$ define $x + y = (x_1 + y_1, \ldots, x_k + y_k)$. Then $+$ is a commutative and associative binary operation in N^k.

Solution

It is easily verified that these properties are carried over from the corresponding properties of addition in N.

EXERCISE 1.1

1. Prove that $A^c - B^c = B - A$
2. Prove that $A \cap (B - C) = (A \cap B) - (A \cap C)$
3. If R is the relation "less than" from $A = \{1, 2, 3, 4, 5\}$ to $B = \{1, 4, 5\}$, write down the set of ordered pairs corresponding to R. Find the inverse relation to R.
4. If R is the relation in $N \times N$ defined by $(a, b)R(c, d)$ if and only if $a + d = b + c$, show that R is an equivalence relation.
5. If $N_7 = \{1, 2, 3, 4, 5, 6, 7\}$, which of the following two is a partition giving rise to an equivalence relation? Why?

   (i) $A_1 = \{1, 3, 5\}$, $A_2 = \{2\}$, $A_3 = \{4, 7\}$

   (ii) $B_1 = \{1, 2, 5, 7\}$, $B_2 = \{3\}$, $B_3 = \{4, 6\}$
6. If $f : A \to B$, $g : B \to C$, are one-to-one or injective functions, show that $g \circ f$ is also one-to-one.
7. If $A = \{a, b, c, d\}$ and f corresponds to the Cartesian product $\{(a, b), (b, d), (c, a), (d, c)\}$ show that f is one-to-one from A onto A. Find f^{-1}.
8. Define the binary operation $\circ$ in N by
   $m \circ n = lcm\ (m, n), m, n \in N$.
   Is the operation commutative and /or associative?

9. Does the table below give a commutative binary operation on the set $\{a, b, c\}$?

o	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

10. Establish the De Morgan's laws stated in section 1.1

11. (i) Prove that if a set has only one element, then it has 2 subsets.
    (ii) If $B \subset A$ and if A has one element more than B, prove that A has twice as many subsets as B.
    (iii) Deduce from these two results that a set with 2 elements has 2^2 subsets, a set with 3 elements has 2^3 subsets and so on. How many subsets does a set with n elements have?

12. Give an example of a map
    (i) which is one-to-one but *not* onto,
    (ii) which is *not* one-to-one but onto,
    (iii) which is neither one-to-one *nor* onto.

13. If A is a non-empty set and $f, g : A \to A$ are such that $f \circ g = g \circ f = I_A$, show that f and g are bijections and that $g = f^{-1}$.

14. For any relation R in a set A, we can define the *inverse* relation R^{-1} by $aR^{-1}b$ if and only if bRa. Prove that
R is symmetric if and only if $R = R^{-1}$

15. In $\mathbf{N} \times \mathbf{N}$, show that the relation defined by $(a, b)R(c, d)$ if and only if $ad = bc$ is an equivalence relation.

16. If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

17. $f, g : \mathbf{R} \to \mathbf{R}$ are defined respectively by $f(x) = x^2 + 3x + 1, g(x) = 2x - 3$, find
    (i) $f \circ g$
    (ii) $g \circ f$
    (iii) $f \circ f$
    (iv) $g \circ g$

18. What is the set $\{x | x \in \mathbf{R}, x^2 = 9 \text{ and } , 2x = 4\}$?

19. Is inclusion of a subset in another, i.e., A$\mathcal{R}$B if and only if A$\subset$B, in the context of a universal set, an equivalence relation in the class of subsets of the universal set? Justify your answer.

20. How many relations are possible from a set A of m elements to another set B of n elements? Why?

21. If $A = \{1, 2, 3\}$, $B = \{4\}$, $C = \{5\}$, then verify that

    (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

    (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

    (iii) $A \times (B - C) = (A \times B) - (A \times C)$

22. '$*$' is a binary operation defined on $\mathbf{Q}$. Find which of the binary operations are commutative

    (i) $a * b = a - b$ for $a, b \in \mathbf{Q}$

    (ii) $a * b = a^2 + b^2$ for $a, b \in \mathbf{Q}$

    (iii) $a * b = a + ab$ for $a, b \in \mathbf{Q}$

    (iv) $a * b = (a - b)^2$ for $a, b \in \mathbf{Q}$

23. '$*$' is a binary operation defined on $\mathbf{Q}$. Find which of the binary operations are associative

    (i) $a * b = a - b$ for $a, b \in \mathbf{Q}$

    (ii) $a * b = \frac{ab}{4}$ for $a, b \in \mathbf{Q}$

    (iii) $a * b = a - b + ab$ for $a, b \in \mathbf{Q}$

    (iv) $a * b = ab^2$ for $a, b \in \mathbf{Q}$

CHAPTER 2

Trigonometric Functions

2.1 Introduction

We shall now begin the study of trigonometry. It is convenient to use trigonometry to measure distances between two landmarks or widths or depths of rivers or heights of mountains, etc.

Trigonometry means the science of measuring triangles. Given some of the sides and angles of a triangle, trigonometry helps us to calculate the remaining sides and angles. The congruence results of geometry tell us that two sides and the included angle (SAS) completely determine the triangle. That is, if we know two of the three sides and the included angle of a triangle, the remaining one side and two angles become fixed. So we should be able to calculate these, but how? School geometry does not tell us how. We have to study trigonometry to be able to do that. In the same way, given three sides of a triangle (SSS), trigonometry will show us how to calculate the angles, etc.

You will be happy to know that the study of trigonometry was first started in India. Elements of the subject can be found even in Rigveda. All the ancient Indian Mathematicians like Aryabhata, Bhaskara I and II and Brahmagupta got important results. All this knowledge first went from India to middle-east and from there to Europe. The Greeks had also started a study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughtout the world.

2.2 Angles

An angle is considered as the figure obtained by rotating a given ray about its end-point. The original ray is called the *initial side* and the ray into which the initial side rotates is called the *terminal side* of the angle. The *sense* of an angle is derived from the direction of rotation of the initial side into the terminal side. If this direction is counter-clockwise, the sense of the angle is said to be positive and if this direction is clockwise then the sense is negative.

The measure of an angle is the amount of rotation required to get to the terminal side from the initial side. We place the vertex of the angle at the centre of a circle of some

fixed radius. Divide the circumference of the circle into 360 equal parts, called degrees.

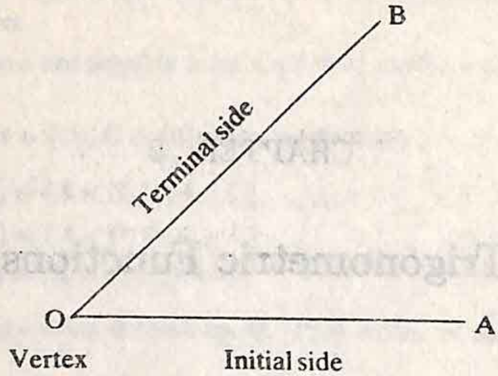


Fig.2.1

The number of degrees on the circumference between the initial and terminal sides of the angle is its degree measure. For additional precision, each degree is subdivided into 60 equal parts, called minutes and each minute is divided into 60 equal parts called seconds. The symbol ° is used to denote degrees, ′ is used to denote minutes and ″ is used to denote seconds.

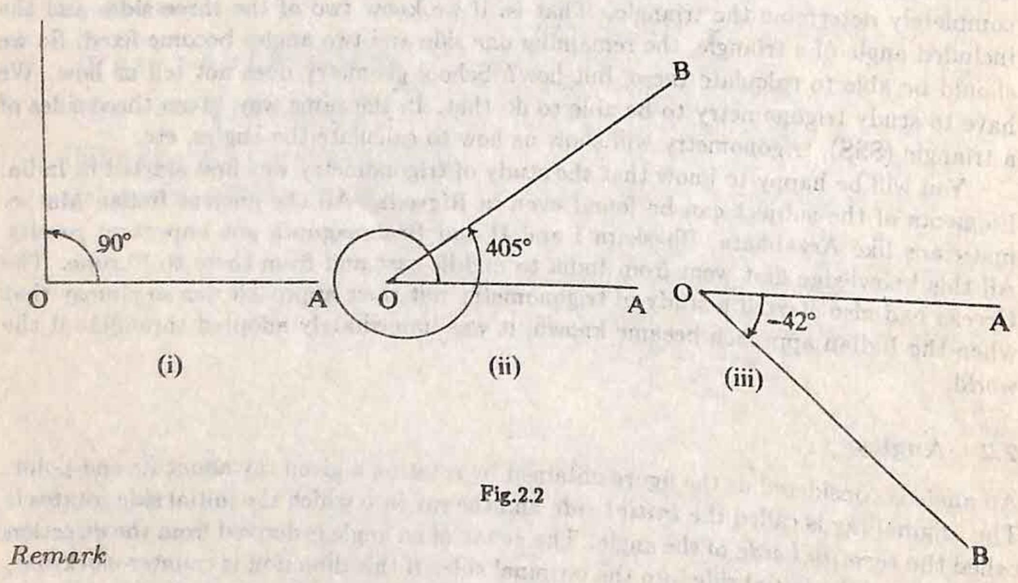


Fig.2.2

Remark

In our discussion in the above paragraph we considered a circle of fixed radius. However, the measure of an angle does not depend on the radius of the circle.

Some of the angles are shown in the following figure. Note that the terminal side has to be rotated so as to have more than one revolution if the corresponding angle is greater than 360°.

Radian Measure

There is another unit of angular measurement called the *radian measure*, which is of particular importance in higher mathematics and its applications. This is based upon the fact that the ratio of the circumference of circle to its diameter is constant. This constant π is an irrational number having the non-recurring decimal expansion $\pi = 3.14159\ldots$; 22/7 is taken as an approximate value of π.

As shown in Fig. 2.3 we place the vertex of the angle at the centre of the circle of radius r. If the length of the arc subtending the angle at the centre is s, the *radian measure* t of the angle is defined to be $\frac{s}{r}$. Note that the length of the arc is taken to be negative if we measure it in the clockwise direction. We again remark that the measure of an angle is independent of the radius of the circle considered because of the fact that the ratio of the circumference of a circle to its diameter is constant.

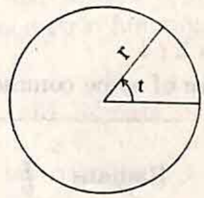


Fig.2.3

The radian measure t of the angle formed by one complete revolution of the initial side is $\frac{2\pi r}{r} = 2\pi$. Thus 2π radians $= 360°$ or π radians $= 180°$. Hence

$$1 \text{ radian} = \frac{180°}{\pi} = 57° 16' \text{ approximately}$$

and $1° = \frac{\pi}{180}$ radian $= 0.01746$ radian appproximately.
Note that the definition $t = \frac{s}{r}$ can be used to find one of s, t and r provided the other two are given.

Example 2.1
Find the length of the arc of a circle of radius 5 cm subtending an angle measuring 45°.

Solution

$$t = \frac{\pi}{180} \times 45 = \frac{\pi}{4}. \text{ Hence } s = r \times t = \frac{5\pi}{4} \text{ cm}$$

Example 2.2

Suppose arcs of the same length in two circles subtend angles of 60° and 75° at the centre. Find the ratio of their radii.

Solution

Let r_1, r_2 denote the radii of the two circles. Now

$$60^\circ = \frac{\pi}{180} \times 60 = \frac{\pi}{3} \text{ and } 75^\circ = \frac{\pi}{180} \times 75 = \frac{5\pi}{12}$$

Let s be the length of the arcs. Then

$$s = \frac{\pi}{3}.r_1 = \frac{5\pi}{12}.r_2$$

Hence $r_1 : r_2 = 5 : 4$

Radian measure of some common angles are given in the following table:

Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Degrees	30°	45°	60°	90°	180°	270°	360°

Note that when an angle is expressed in radians, the word 'radians' is often omitted. Thus $\pi = 180^\circ$ is really a short form of writing π radians $= 180^\circ$.

EXERCISE 2.1

1. Find the radian measure corresponding to the following degree measures
   (a) 15° (b) $-22^\circ\ 30'$ (c) 340° (d)420°
2. Find the degree measure corresponding to the following radian measures
   (a) $\frac{1}{4}$ (b) -2 (c) $\frac{7\pi}{3}$ (d) $\frac{5\pi}{6}$
3. Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15°.
4. In a circle of diameter 40 cm. the length of a chord is 20 cm. Find the length of minor arc of the chord.
5. A wheel makes 180 revolutions in one minute. Through how many radians does it turn in one second?

6. Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an arc of 11 cm (use $\pi = \frac{22}{7}$).

7. Find the angle through which a pendulum swings if its length is 50 cm and the tip describes an arc of length (a) 10cm (b) 16 cm (c) 26 cm. (use $\pi = \frac{22}{7}$).

8. Find the angle between the minute hand of a clock and the hour hand when the time is 7.20.

2.3 Circular Functions or Trigonometric Functions

Let θ be the angle XOC as shown in Fig. 2.4 and let $P(x, y)$ be any point other than O on its terminal side OC. Let length of $OP = r$. We take always r to be > 0. We define the following functions known as circular or trigonometric functions. These are also called trigonometric ratios.

$$\text{sine } \theta = \sin\theta = \frac{y}{r}$$

$$\text{cosine } \theta = \cos\theta = \frac{x}{r}$$

$$\text{tangent } \theta = \tan\theta = \frac{y}{x} \quad (\theta \neq \frac{\pi}{2})$$

$$\text{cosecant } \theta = \text{cosec}\,\theta = \frac{r}{y} \quad (\theta \neq 0)$$

$$\text{secant } \theta = \sec\theta = \frac{r}{x} \quad (\theta \neq \frac{\pi}{2})$$

$$\text{cotangent } \theta = \cot\theta = \frac{x}{y}(\theta \neq 0)$$

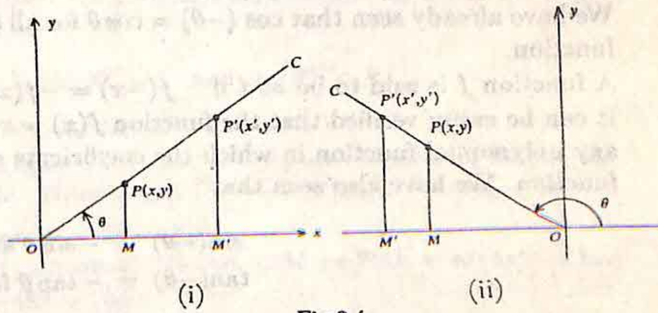


Fig 2.4

Note that these ratios may be positive or negative depending on x and/or y.

We observe that the above functions depend only on the value of the angle θ and not on the point P chosen on the terminal side of the angle θ. For example, if we take another point $P'(x', y')$ on OC with $OP' = r'$ then considering similar triangles we obtain

$$\frac{y}{r} = \frac{y'}{r'}, \frac{x}{r} = \frac{x'}{r'}, \frac{y}{x} = \frac{y'}{x'}$$

This is also true if the terminal side coincides with one of the axes with the only difference that if it coincides with x-axis, then cosec and cot are not defined while in case it coincides with y-axis, then sec and tan are not defined. From the definition and Fig. 2.5, it is clear that

$$\sin\theta = \sin(\theta + 2\pi) \quad \sin(-\theta) = -\sin\theta$$
$$\cos\theta = \cos(\theta + 2\pi) \quad \cos(-\theta) = \cos\theta$$
$$\tan\theta = \tan(\theta + 2\pi) \quad \tan(-\theta) = -\tan\theta$$

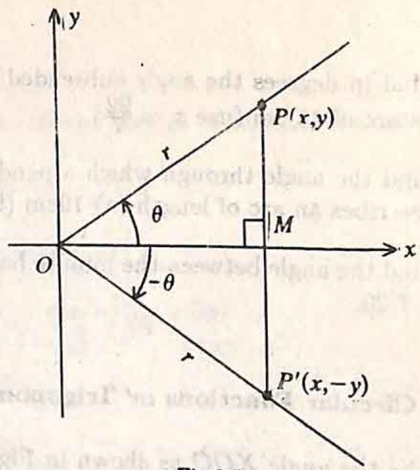


Fig.2.5

At this state, let us notice two important properties of trigonometric functions. Let us briefly introduce the notion of even and odd functions and that of periodic functions. A function f is said to be even if $f(x) = f(-x)$ for all x.

A simple example of an even function is a constant function.
Any polynomial function $p(x) = a_0 + a_1x^2 + a_2x^4 + \ldots + a_nx^{2n}$ (in which there are only even powers of x) is an even function.
We have already seen that $\cos(-\theta) = \cos\theta$ for all θ. Thus cosine function is also an even function.
A function f is said to be *odd* if $f(-x) = -f(x)$ for all x.
It can be easily verified that the function $f(x) = x$, $f(x) = x^3$ are odd functions. In fact any polynomial function in which the coefficients of even powers of x are zero is an odd function. We have also seen that

$$\sin(-\theta) = -\sin\theta \text{ and}$$
$$\tan(-\theta) = -\tan\theta \text{ for all } \theta.$$

Thus sine and tangent functions are also odd.

The property of functions being even or odd is very useful in the study of such functions. It also helps in drawing graph of such functions as once we draw the graph for $x \geq 0$, we can complete the graph of f for all x easily.

The other important property of trigonometric functions which we want to observe now is that of periodicity. A function f is said to be *periodic* if there exists a real $T > 0$ such that $f(x+T) = f(x)$ for all x. We have already noted that

$$\sin(\theta + 2\pi) = \sin\theta$$

$$\text{and } \cos(\theta + 2\pi) = \cos\theta$$

Thus sine and cosine functions are examples of periodic functions. If a function f is periodic then the smallest $T > 0$, if it exists, such that

$$f(x+T) = f(x) \text{ for all } x\,.$$

is called the period of the function. It can be easily seen that the period of the sine and cosine functions is 2π. We shall see later that the period of the tangent function is π. It is interesting to note that a constant function f is periodic as $f(x+T) = f(x)$ for all $T > 0$, however it does not have a period because there is no *smallest* $T > 0$ for which the relation holds.

The periodicity of a function is also very useful concept. In particular, it follows that the graph of such a function is completely known once we know it over an interval whose length is equal to the period of function. The periodicity of trigonometric functions helps us to compute the value of these functions for large θ. For example,

$$\sin 785^\circ = \sin\,(2 \times 360^\circ + 65^\circ) = \sin 65^\circ$$

$$\text{and } \cos\,(-2070^\circ) = \cos\,(-2070^\circ + 6 \times 360^\circ) = \cos 90^\circ$$

Values of Trigonometric Functions for the Angles 0°, 30°, 45°, 60°, 90°.
From the definition of trigonometric functions it follows that the values of all trigonometric functions for given angle are known, once we find the sine and cosine of the angle.

It is clear from the definition that

$$\sin 0^\circ = 0,\ \cos 0^\circ = 1,\ \sin 90^\circ = 1, \text{ and } \cos 90^\circ = 0.$$

Let us calculate the values of these functions for 30°, 45°, 60°:

In Fig. 2.6, $PM = MP'$. The two triangles OPM and $OP'M$ are congruent. Hence, the triangle OPP' is an equilateral triangle. Therefore, if $PP' = 2a$, then $OP = 2a$, $PM = a$ and $OM = a\sqrt{3}$. Hence, $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

In Fig. 2.7, angles POM and OPM are equal. Hence, $OM = PM = a$(say). Then $OP = a\sqrt{2}$ and $\sin 45^\circ = \dfrac{1}{\sqrt{2}}$ and $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$.

In Fig. 2.8, $OM = MM'$, the triangles OPM and $M'PM$ are congruent and the triangle OPM' is equilateral. Hence, if $OM = a$, then $OP = 2a$ and $PM = a\sqrt{3}$. Therefore, $\sin 60^\circ = \frac{\sqrt{3}}{2}$ and $\cos 60^\circ = \frac{1}{2}$.

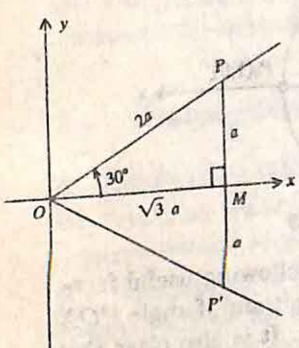


Fig 2.6

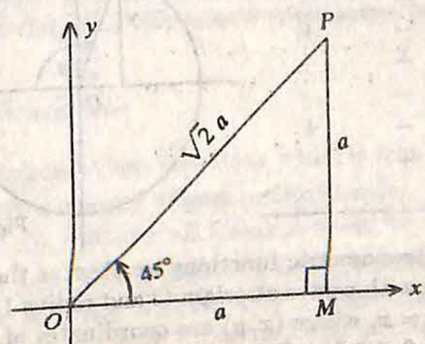


Fig 2.7

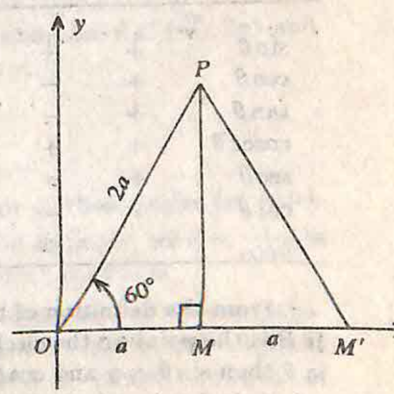


Fig 2.8

Thus we have the following table:

θ	$0°$	$30°$	$45°$	$60°$	$90°$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

Notational Convention

Since angles are measured either in radians or degrees, we adopt the convention that whenever we write $\cos\theta°$, we mean the cosine of the angle whose degree measure is $\theta°$ (and similarly for other ratios) and whenever we write $\cos\beta$, we mean the cosine of the angle whose measure in radians is β. If no such convention used, it should be clear from the context, which meaning is being used.

Signs of the Trigonometric Functions

The signs of the trigonometric functions depend on the quadrant in which the terminal arm of the angle lies. Thus $\sin\theta = \frac{y}{r}$ has the sign of y as r is always positive. Therefore $\sin\theta$ is taken as positive if the *angle* is in first or second quadrant, while it is negative for the *angles* in the third or fourth quadrant. Similarly $\cos\theta$ is positive in the first and fourth quadrants and negative in the remaining two quadrants. In fact we have the following table:

	I	II	III	IV
$\sin\theta$	+	+	−	−
$\cos\theta$	+	−	−	+
$\tan\theta$	+	−	+	−
$\operatorname{cosec}\theta$	+	+	−	−
$\sec\theta$	+	−	−	+
$\cot\theta$	+	−	+	−

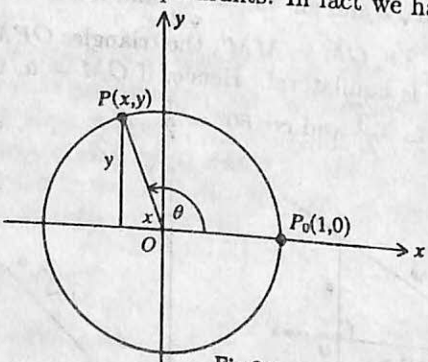


Fig.2.9

From the definition of trigonometric functions we observe the following useful facts. If P is the point on the circle with centre at origin O and radius 1 unit and if angle POX is θ then $\sin\theta = y$ and $\cos\theta = x$, where (x, y) are coordinates of P. It is also clear that $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$ for all θ.

In the first quadrant as the angle increases from $0°$ to $90°$, $\sin\theta$ increases from 0 to 1. In the second quadrant as θ increases from $90°$ to $180°$, $\sin\theta$ decreases from 1 to 0. In the third quadrant as θ increases from $180°$ to $270°$, $\sin\theta$ decreases from 0 to -1 and finally in the fourth quadrant $\sin\theta$ increases from -1 to 0 as θ increases from $270°$ to $360°$. In fact we have the following table:

	I quadrant		*II qudrant*
sine	increases from 0 to 1	sine	decreases from 1 to 0
cosine	decreases from 1 to 0	cosine	decreases from 0 to -1
tangent	increases from 0 to ∞	tangent	increases from $-\infty$ to 0
cotangent	decreases from ∞ to 0	cotangent	decreases from 0 to $-\infty$
secant	increases from 1 to ∞	secant	increases from $-\infty$ to -1
cosecant	decreases from ∞ to 1	cosecant	increases from 1 to ∞

	III quadrant		*IV quadrant*
sine	decreases from 0 to -1	sine	increases from -1 to 0
cosine	increases from -1 to 0	cosine	increases from 0 to 1
tangent	increases from 0 to ∞	tangent	increases from $-\infty$ to 0
cotangent	decreases from ∞ to 0	cotangent	decreases from 0 to $-\infty$
secant	decreases from -1 to $-\infty$	secant	decreases from ∞ to 1
cosecant	increases from $-\infty$ to -1	cosecant	decreases from -1 to $-\infty$

Remark

In the above table we see the symbol ∞. Observe that ∞ is not a real number and is just a symbol. Statement like $\tan\theta$ increases from 0 to ∞ for $\theta \in (0, \frac{\pi}{2})$ simply means that $\tan\theta$ increases as θ increases in the interval $(0, \frac{\pi}{2})$ and assumes arbitrarily large positive values as θ increases to $\frac{\pi}{2}$. Similarly, to say that cosecant decreases from -1 to $-\infty$ in the fourth quadrant means that $\operatorname{cosec}\theta$ is a *decreasing function* for $\theta \in (\frac{3\pi}{2}, 2\pi)$ and assumes arbitrarily large negative values as θ approaches 2π.

2.4 Trigonometric Identities

An equation involving trigonometric functions which is true for all those angles for which the functions are defined is called a trigonometric identity. For example, $\sec\theta = \dfrac{1}{\cos\theta}$ is a trigonometric identity. It holds for all θ except those for which $\cos\theta = 0$.

An equation of the form $\sin\theta = \cos\theta$ is a trigonometric equation but not a trigonometric identity because it is not true for all θ. Trigonometric identities and solutions of trigonometric equations are very important and are useful in various problems of engineering and science.

Fundamental Identities

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

All of the above identities are very easy to prove and the proofs are left as an exercise. From these we also observe that given one of the six trigonometric functions, all others can be found numerically and their signs can be found by seeing in which quadrant the angle lies.

Example 2.3

Given $\cot\theta = \frac{12}{5}$, θ in quadrant III, find the values of the other five functions.

Solution

$$\tan\theta = \frac{5}{12}, \qquad \sec^2\theta = 1 + \tan^2\theta = 1 + \frac{25}{144} = \frac{169}{144}$$

Now in quadrant III, $\sin\theta$, $\cos\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ are all negative. Therefore

$$\sec\theta = -\frac{13}{12}, \qquad \cos\theta = -\frac{12}{13}, \qquad \sin\theta = \tan\theta\cos\theta = \frac{5}{12}\left(\frac{-12}{13}\right) = \frac{-5}{13},$$

$$\operatorname{cosec}\theta = \frac{-13}{5}$$

Example 2.4

Prove that $\sqrt{\frac{1-\sin A}{1+\sin A}} = \sec A - \tan A$

Solution

$$\text{L.H.S. } \frac{\sqrt{1-\sin A}}{\sqrt{1+\sin A}}\,\frac{\sqrt{1-\sin A}}{\sqrt{1-\sin A}} = \frac{1-\sin A}{\sqrt{1-\sin^2 A}} = \frac{1-\sin A}{\cos A} = \sec A - \tan A$$

$$= \text{R.H.S.}$$

Example 2.5
Prove that
$$\frac{\sin\theta}{1-\cos\theta}+\frac{\tan\theta}{1+\cos\theta}=\sec\theta\operatorname{cosec}\theta+\cot\theta$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{\sin\theta}{1-\cos\theta}+\frac{\tan\theta}{1+\cos\theta}\\
&= \frac{\sin\theta+\sin\theta\cos\theta+\tan\theta-\tan\theta\cos\theta}{1-\cos^2\theta}\\
&= \frac{\sin\theta+\sin\theta\cos\theta+\tan\theta-\sin\theta}{\sin^2\theta}\\
&= \frac{\sin\theta\cos\theta+\tan\theta}{\sin^2\theta}\\
&= \cot\theta+\sec\theta\operatorname{cosec}\theta=\text{R.H.S.}
\end{aligned}$$

Example 2.6
Prove that
$$\frac{\tan A+\sec A-1}{\tan A-\sec A+1}=\frac{1+\sin A}{\cos A}$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{\tan A+\sec A-1}{\tan A-\sec A+1}=\frac{\sin A+1-\cos A}{\sin A-1+\cos A}\\
&= \frac{(\sin A+1-\cos A)}{(\sin A+\cos A-1)}\times\frac{(\sin A+1+\cos A)}{(\sin A+1+\cos A)}\\
&= \frac{(\sin A+1)^2-\cos^2 A}{(\sin A+\cos A)^2-1}=\frac{1+2\sin A+\sin^2 A-\cos^2 A}{\sin^2 A+\cos^2 A+2\sin A\cos A-1}\\
&= \frac{2\sin A+2\sin^2 A}{2\sin A\cos A}\\
&= \frac{1+\sin A}{\cos A}=\text{R.H.S.}
\end{aligned}$$

EXERCISE 2.2

Find the values of the other five trigonometric functions in each of the following problems:

1. $\cos\theta=-\frac{1}{2}$, θ in quadrant II

2. $\sin\theta = \frac{3}{5}$, θ in quadrant I

3. $\tan\theta = \frac{3}{4}$, θ in quadrant III

Prove the following trigonometric identities:

4. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

5. $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

6. $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$

7. $\dfrac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \operatorname{cosec}^2\theta$

8. $\dfrac{\operatorname{cosec}\theta}{\cot\theta + \tan\theta} = \cos\theta$

9. $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$

10. $\dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$

11. $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

12. $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

13. $2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta = \cot^4\theta - \tan^4\theta$

2.5 Cosine of the Difference of Two Angles

We begin by establishing a formula for the cosine of the difference of two angles in terms of sines and cosines of the individual angles. We shall see that this helps us in proving several other important identities.

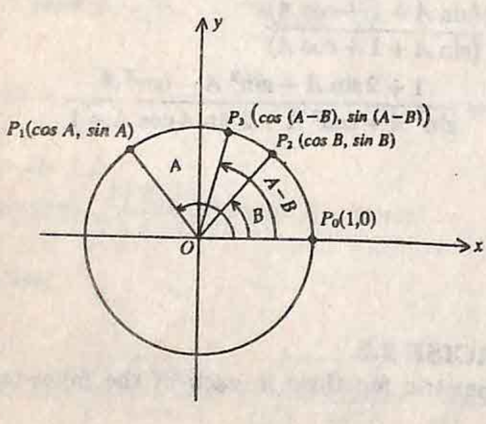


Fig 2.10 (i)

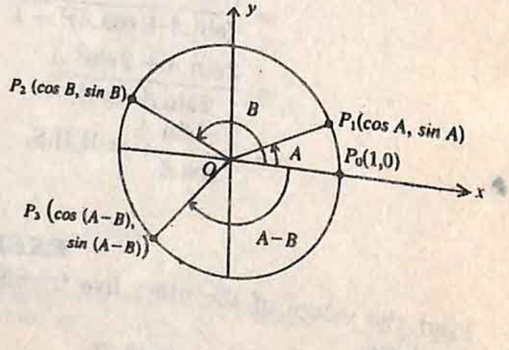


Fig 2.10 (ii)

Recall that the terminal side of any angle cuts the circle with centre at O and unit radius at a point whose coordinates are respectively the cosine and sine of the angle. In Fig. 2.10 OP_1 and OP_2 are the terminal sides of the angles A and B respectively and OP_3 has been drawn to be the terminal side of the angle $A-B$. It is now clear that the chords P_0P_3 and P_1P_2 subtend the central angles of same size and hence are equal in length. Therefore, we obtain $[\cos(A-B)-1]^2+\sin^2(A-B) = (\cos B-\cos A)^2+(\sin B-\sin A)^2$. This on simplification yields

$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

The method of proof of the above formula applied to all values of A and B. Hence, the formula holds for al! angles A and B, positive, zero or negative.

Example 2.7
Find the values of $\cos 15°$ and $\cos 75°$.

Solution

$$\begin{aligned}\cos 15° = \cos(45° - 30°) &= \cos 45° \cos 30° + \sin 45° \sin 30° \\ &= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\text{Similarly,}\quad \cos 75° &= \cos(45° - (-30°)) \\ &= \cos 45° \cos 30° + \sin 45° \sin(-30°) \\ &= \cos 45° \cos 30° - \sin 45° \sin 30° \\ &= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

Ratios of Complementary Angle

$$\begin{aligned}\cos(90° - \theta) &= \cos 90° \cos\theta + \sin 90° \sin\theta \\ &= \sin\theta \\ \sin(90° - \theta) &= \cos(90° - (90° - \theta)) = \cos\theta\end{aligned}$$

Hence, we have

$$\begin{aligned}\sin(90° - \theta) &= \cos\theta & \operatorname{cosec}(90° - \theta) &= \sec\theta \\ \cos(90° - \theta) &= \sin\theta & \sec(90° - \theta) &= \operatorname{cosec}\theta \\ \tan(90° - \theta) &= \cot\theta & \cot(90° - \theta) &= \tan\theta\end{aligned}$$

From the above, replacing θ by $-\theta$ and recalling the formulas on section no. 2.3, we obtain

$$\begin{aligned}
\sin(90^\circ+\theta) &= \cos\theta & \operatorname{cosec}(90^\circ+\theta) &= \sec\theta\\
\cos(90^\circ+\theta) &= -\sin\theta & \sec(90^\circ+\theta) &= -\operatorname{cosec}\theta\\
\tan(90^\circ+\theta) &= -\cot\theta & \cot(90^\circ+\theta) &= -\tan\theta
\end{aligned}$$

Formulas for Functions of Sum and Difference of Two Angles

$$\begin{aligned}
\cos(A+B) &= \cos(A-(-B))\\
&= \cos A\cos(-B)+\sin A\sin(-B)\\
&= \cos A\cos B-\sin A\sin B\\
\sin(A-B) &= \cos(90^\circ-(A-B))\\
&= \cos((90^\circ-A)+B)\\
&= \cos(90^\circ-A)\cos B-\sin(90^\circ-A)\sin B\\
&= \sin A\cos B-\cos A\sin B\\
\sin(A+B) &= \sin(A-(-B))\\
&= \sin A\cos(-B)-\cos A\sin(-B)\\
&= \sin A\cos B+\cos A\sin B\\
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)}\\
&= \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}\\
&= \frac{\tan A+\tan B}{1-\tan A\tan B} \quad \text{(divide numerator and denominator by } \cos A\cos B\text{)}
\end{aligned}$$

Similar computations yield

$$\begin{aligned}
\cot(A+B) &= \frac{\cot A\cot B-1}{\cot B+\cot A}\\
\tan(A-B) &= \frac{\tan A-\tan B}{1+\tan A\tan B}\\
\cot(A-B) &= \frac{\cot A\cot B+1}{\cot B-\cot A}
\end{aligned}$$

Using above formulas, we get

$$\begin{aligned}
\sin(180^\circ-\theta) &= \sin\theta & \operatorname{cosec}(180^\circ-\theta) &= \operatorname{cosec}\theta\\
\cos(180^\circ-\theta) &= -\cos\theta & \sec(180^\circ-\theta) &= -\sec\theta\\
\tan(180^\circ-\theta) &= -\tan\theta & \cot(180^\circ-\theta) &= -\cot\theta\\
\sin(180^\circ+\theta) &= -\sin\theta & \operatorname{cosec}(180^\circ+\theta) &= -\operatorname{cosec}\theta\\
\cos(180^\circ+\theta) &= -\cos\theta & \sec(180^\circ+\theta) &= -\sec\theta\\
\tan(180^\circ+\theta) &= \tan\theta & \cot(180^\circ+\theta) &= \cot\theta
\end{aligned}$$

From this we conclude that the period of the tangent function is π. All these can be directly deduced from the definitions given in section 2.3.
Also,

$$\begin{aligned}
2\sin A\cos B &= \sin(A+B)+\sin(A-B)\\
2\cos A\sin B &= \sin(A+B)-\sin(A-B)\\
2\cos A\cos B &= \cos(A+B)+\cos(A-B)\\
2\sin A\sin B &= \cos(A-B)-\cos(A+B)
\end{aligned}$$

These are called product formulas.
We also have the following sum formulas.

$$\begin{aligned}
\sin A+\sin B &= \sin\left(\frac{A+B}{2}+\frac{A-B}{2}\right)+\sin\left(\frac{A+B}{2}-\frac{A-B}{2}\right)\\
&= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sin A-\sin B &= 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\\
\cos A+\cos B &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\\
\cos A-\cos B &= -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}
\end{aligned}$$

Values of Functions at $2A$, $\frac{1}{2}A$ *and* $3A$

$$\begin{aligned}
\sin 2A=\sin(A+A) &= 2\sin A\cos A\\
\cos 2A=\cos(A+A) &= \cos^2 A-\sin^2 A\\
&= 2\cos^2 A-1\\
&= 1-2\sin^2 A\\
\tan 2A &= \tan(A+A)\\
&= \frac{2\tan A}{1-\tan^2 A}
\end{aligned}$$

From the above $2\cos^2 A=1+\cos 2A$ and
$2\sin^2 A=1-\cos 2A$

Hence, replacing A by $\frac{A}{2}$, in the above formula, we get

$$\sin\frac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{2}}$$
$$\cos\frac{1}{2}A = \pm\sqrt{\frac{1+\cos A}{2}}$$
$$\tan\frac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\begin{aligned}
\text{Also } \tan\frac{1}{2}A &= \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A}\\
\text{Again } \sin 3A &= \sin(A+2A)\\
&= \sin A\cos 2A + \cos A \sin 2A\\
&= \sin A(1-2\sin^2 A) + 2\sin A(1-\sin^2 A)\\
&= 3\sin A - 4\sin^3 A.
\end{aligned}$$

Similarly we can show that

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$. The student is advised to prove these.

Values of sine and cosine for Some Special Angles

We have already found the values of $\cos 15°$ and $\cos 75°$. Recall that once we know the values of $\cos 15°$ and $\cos 75°$, it is easy to find the values of $\sin 15°$ and $\sin 75°$. Thus

$$\sin 15° = \sin(90° - 75°) = \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

and $\sin 75° = \cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

Trigonometric Functions of an Angle of 18°

Let $\theta = 18°$. Then $2\theta = 90° - 3\theta$

Therefore

$$\begin{aligned}
\sin 2\theta &= \cos 3\theta\\
\text{or } 2\sin\theta\cos\theta &= 4\cos^3\theta - 3\cos\theta.
\end{aligned}$$

Since $\cos\theta \neq 0$, we get

$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta$

or $4\sin^2\theta + 2\sin\theta - 1 = 0$

Hence $\sin\theta = \dfrac{-2 \pm \sqrt{4+16}}{2 \times 4} = \dfrac{-1 \pm \sqrt{5}}{4}$

Since $\theta = 18^\circ, \sin\theta > 0$. Therefore $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$

Also $\cos 18^\circ = \sqrt{1 - \sin^2 18^\circ} = \sqrt{1 - \dfrac{6 - 2\sqrt{5}}{16}} = \sqrt{\dfrac{10 + 2\sqrt{5}}{16}}$

Hence, $\cos 18^\circ = \dfrac{\sqrt{10 + 2\sqrt{5}}}{4}$.

Now we can easily find $\cos 36^\circ$, $\sin 36^\circ$.

$$\begin{aligned} \cos 36^\circ = 1 - 2\sin^2 18^\circ &= 1 - \frac{2(6 - 2\sqrt{5})}{16} \\ &= \frac{2 + 2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4}. \end{aligned}$$

$$\text{Hence, } \cos 36^\circ = \frac{\sqrt{5}+1}{4}.$$

$$\begin{aligned} \text{Also } \sin 36^\circ &= \sqrt{1 - \cos^2 36^\circ} = \sqrt{1 - \frac{6 + 2\sqrt{5}}{16}} \\ &= \frac{\sqrt{10 - 2\sqrt{5}}}{4} \end{aligned}$$

Example 2.8

Prove that

$$\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$$

Solution

$\cos 15^\circ = \sin(90^\circ - 15^\circ) = \sin 75^\circ$

$\cos 105^\circ = \cos(90^\circ + 15^\circ) = -\sin 15^\circ$

Therefore, $\cos 105^\circ + \cos 15^\circ = \sin 75^\circ - \sin 15^\circ$

Example 2.9

Prove that

$$\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$$

Solution

L.H.S. $= \dfrac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y}$

Dividing the numerator and denominator by $\cos x \cos y$, we get that

$$\text{L.H.S.} = \frac{\tan x - \tan y}{\tan x + \tan y} = \text{R.H.S.}$$

Example 2.10
Prove that

$$\frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A$$

Solution

$$\text{L.H.S.} = \frac{2\cos 4A\ \sin A}{2\cos 4A\ \cos A} = \tan A = \text{R.H.S.}$$

Example 2.11
Prove that

$$\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{1}{2}\left[2\cos 2\theta\cos\frac{\theta}{2} - 2\cos 3\theta\cos\frac{9\theta}{2}\right] \\
&= \frac{1}{2}\left[\cos\left(2\theta+\frac{\theta}{2}\right) + \cos\left(2\theta-\frac{\theta}{2}\right) - \cos\left(3\theta+\frac{9\theta}{2}\right) - \cos(3\theta - \frac{9\theta}{2})\right] \\
&= \frac{1}{2}\left[\cos\frac{5\theta}{2} + \cos\frac{3\theta}{2} - \cos\frac{15\theta}{2} - \cos\frac{3\theta}{2}\right] \\
&= \frac{1}{2}\left[\cos\frac{5\theta}{2} - \cos\frac{15\theta}{2}\right] \\
&= \sin\frac{5\theta+15\theta}{4}\sin\frac{15\theta-5\theta}{4} \\
&= \sin 5\theta \sin\frac{5\theta}{2} = \text{R.H.S.}
\end{aligned}$$

Example 2.12
Prove that

$$\frac{\sin 2A}{1+\cos 2A} = \tan A$$

Solution

$$\text{L.H.S.} = \frac{2\sin A\cos A}{2\cos^2 A} = \tan A = \text{R.H.S}$$

Example 2.13
Prove that

$$\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$$

Solution

$$\begin{aligned}\text{L.H.S.} &= 2\sin 2A\cos 2A\\ &= 4\sin A\cos A(\cos^2 A-\sin^2 A)\\ &= 4\sin A\cos^3 A-4\cos A\sin^3 A\\ &= \text{R.H.S.}\end{aligned}$$

Example 2.14
Find the value of $\tan 22°30'$

Solution

$$\text{Let } \theta = 45° \quad \text{Then } \frac{\theta}{2} = 22°\,30'$$

$$\begin{aligned}\text{Now } \tan\frac{\theta}{2} &= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\\ &= \frac{\sin\theta}{1+\cos\theta} = \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}+1}\\ &= \frac{1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1} = \sqrt{2}-1\end{aligned}$$

Hence, $\tan 22°30' = \sqrt{2}-1$

EXERCISE 2.3

1. If $\sin\alpha = \frac{15}{17}$ and $\cos\beta = \frac{12}{13}$, find the values of $\sin(\alpha+\beta)$, $\cos(\alpha-\beta)$ and $\tan(\alpha+\beta)$.

2. Prove that

$$\cos(45°-A)\cos(45°-B)-\sin(45°-A)\sin(45°-B)=\sin(A+B)$$

3. Show that

$$\sin 105° + \cos 105° = \cos 45°$$

4. Prove that

$$\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A=\cos A$$

5. Prove that

$$\frac{\tan(45^\circ+x)}{\tan(45^\circ-x)}=\left[\frac{1+\tan x}{1-\tan x}\right]^2$$

6. $\dfrac{\sin A+\sin 3A}{\cos A+\cos 3A}=\tan 2A$

7. $\dfrac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4\cos 2\theta\cos 4\theta$

8. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\left(\dfrac{A+B}{2}\right)$

9. $\cos 4x=1-8\sin^2 x\cos^2 x$

10. $\tan 4\theta=\dfrac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}$

11. $(\sin 3A+\sin A)\sin A+(\cos 3A-\cos A)\cos A=0$

12. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

Find sine, cosine and tangent of $\frac{x}{2}$ if $0\le x\le 2\pi$ in the following problems:

13. $\tan x=-\frac{4}{3}$, x in quadrant II

14. $\cos x=-\frac{1}{3}$, x in quadrant III

15. $\sin x=\frac{1}{4}$, x in quadrant II

16. Find $\sin 7\frac{1}{2}^\circ$, $\cos 7\frac{1}{2}^\circ$ and $\tan 11\frac{1}{4}^\circ$.

Prove that

17. $\sin^2 72^\circ-\sin^2 60^\circ=\dfrac{\sqrt{5}-1}{8}$

18. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}=\frac{5}{16}$

Prove that

19. $\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}=-\frac{1}{2}$

20. $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=4\cos^2\dfrac{\alpha-\beta}{2}$

21. $\cos^2 A+\cos^2(A+120^\circ)+\cos^2(A-120^\circ)=\dfrac{3}{2}$

22. $\dfrac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$

23. $\sin 3A+\sin 2A-\sin A=4\sin A\cos\dfrac{A}{2}\cos\dfrac{3A}{2}$

24. $\cos 6A=32\cos^6 A-48\cos^4 A+18\cos^2 A-1$

25. $\tan 3A\tan 2A\tan A=\tan 3A-\tan 2A-\tan A$

2.6 Tables of Trigonometric Functions

In order to deal with many problems in trigonometry, it is necessary to find the values of trigonometric functions for various angles. Trigonometric functions of any angle can be computed to any *desired degree of accuracy*. Tables are available which give approximate values of the sine and tangent of angles from 0° to 90°. The values of $\cos\theta$ and $\cot\theta$ can also be easily read out by using formulas like $\sin(90° - \theta) = \cos\theta, \tan(90° - \theta) = \cot\theta$; $\sec\theta$ and $\operatorname{cosec}\theta$ can be found out by noticing that $\sec\theta = \frac{1}{\cos\theta}, \operatorname{cosec}\theta = \frac{1}{\sin\theta}$. There are tables which give the values of six trigonometric functions of angles from 0° to 90° For angles larger than 90° we use various formulas to reduce the value of trigonometric function to the numerical value of some trigonometric function of an angle between 0° and 90°. For example $\sin 124° = \sin(90° + 34°) = \cos 34°$. If sine of some angle is not given in the table we can use linear interpolation to find its value. We illustrate these by some examples.

Example 2.15
Find $\cot 131°20'$.

Solution
First we observe

$$\cot 131°20' = -\tan 41°20'$$

We are using $\cot(90° + \theta) = -\tan\theta$
From the table, we see that $\tan 41°20' = .8796$.
Hence $\cot 131°20' = -.8796$

Example 2.16
Find the angle θ if $\sin\theta = .7071$

Solution
In the table of sines, we see that $\sin 45° = .7071$
Hence, $\theta = 45°$

Example 2.17
Find the value of $\sin 23°26'$.

Solution
From the table we see that

$$\sin 23°20' = .3961$$

and $\sin 23°30' = .3987$ $\therefore$ difference $= 0.0026$
difference on $10'$ is $.0026$.

$$\text{Hence, the difference for } 6' = \frac{6}{10} \times .0026 = .00156$$
$$= .0016 \text{ (approx.)}$$
$$\text{Hence, } \sin 23°26' = .3961 + .0016$$
$$= .3977$$

Example 2.18
Find θ if $\cot\theta = .5750$.

Solution

$\tan(90° - \theta) = \cot\theta = .5750$

From table we see $.5735 = \tan 29°50'$

and $.5774 = \tan 30°$

difference $= .0039$ Also $.5750 - .5735 = .0015$

For $.0039$ difference angle is $10'$.
For $.0015$ difference angle is $\frac{10 \times 15}{39} = 4'$ approx.
Hence $90° - \theta = 29°54'$. Therefore $\theta = 60°6'$.

EXERCISE 2.4

1. Find the following:
   (i) $\cos 20°10'$ (ii) $\sin 48°$ (iii) $\tan 54°30'$ (iv) $\cot 33°40'$
2. Find the angle $\theta, 0° \le \theta \le 90°$, if
   (i) $\sin\theta = 0.5373$ (ii) $\cos\theta = .0087$
   (iii) $\tan\theta = 34.37$ (iv) $\cot\theta = 3.018$
3. Find the following:
   (i) $\sin 34°22'$ (ii) $\cos 64°34'$
   (iii) $\tan 42°6'$ (iv) $\cot 47°26'$
4. Find θ where
   (i) $\sin\theta = .5240$ (ii) $\cos\theta = .0424$
   (iii) $\cot\theta = 1.246$ (iv) $\tan\theta = .1362$

2.7 Graphs of Trigonometric Functions

We have already seen that all trigonometric functions are periodic. For example, the period of the sine and cosine functions is 2π, while that of the tangent functions is π. Often, we have to deal with functions of the form $\sin ax$, $c\cos(ax + b)$ and so on. These

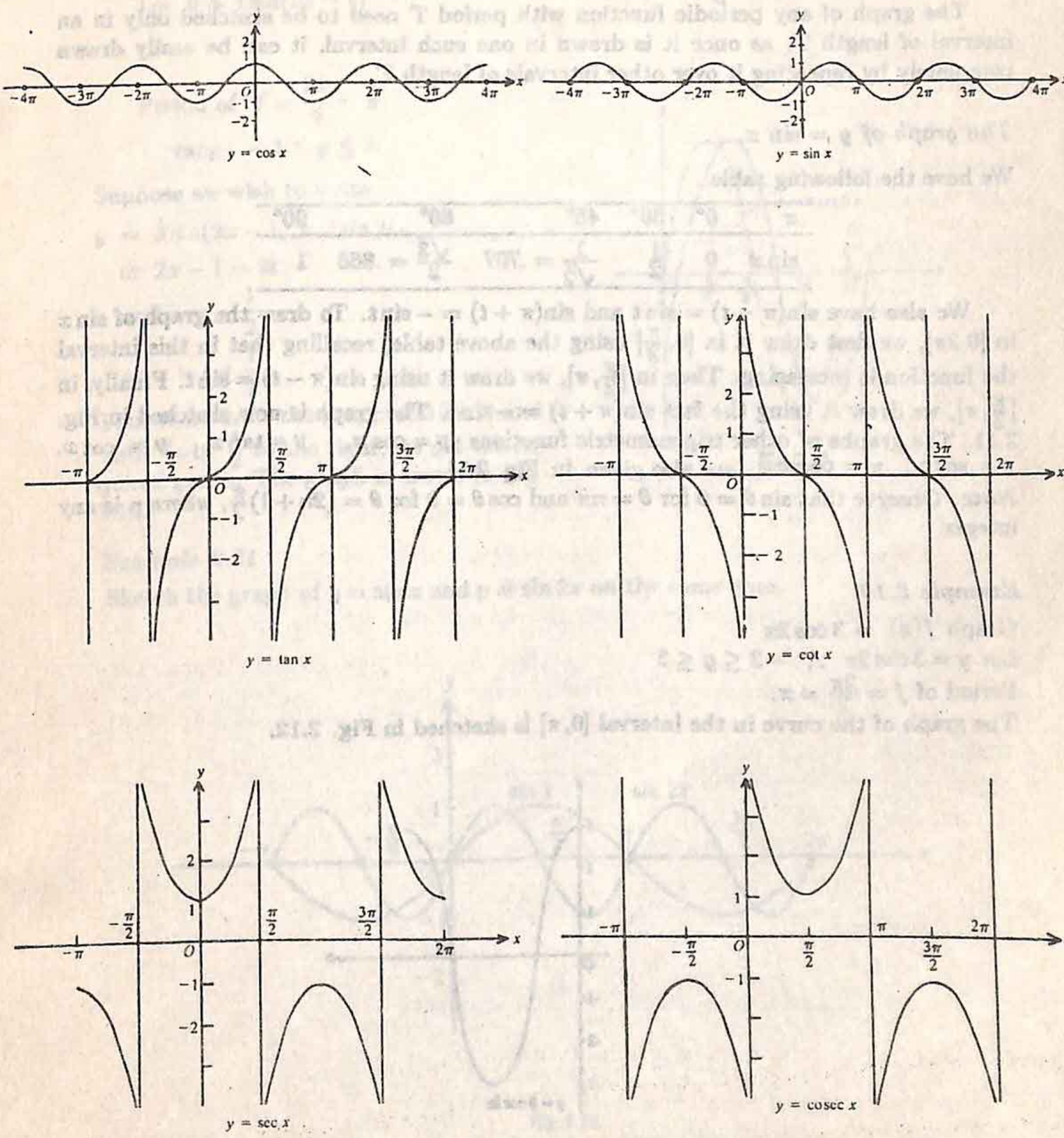


Fig. 2.11

functions are also periodic with period $\frac{2\pi}{a}$ as can be easily verified. For example, the period of $5\sin(3x+4)$ is $\frac{2\pi}{3}$.

The graph of any periodic function with period T need to be sketched only in an interval of length T, as once it is drawn in one such interval, it can be easily drawn completely by repeating it over other intervals of length T.

The graph of $y = \sin x$.

We have the following table

x	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = .707$	$\frac{\sqrt{3}}{2} = .866$	1

We also have $\sin(\pi - t) = \sin t$ and $\sin(\pi + t) = -\sin t$. To draw the graph of $\sin x$ in $[0,2\pi]$, we first draw it in $[0, \frac{\pi}{2}]$ using the above table, recalling that in this interval the function is increasing. Then in $[\frac{\pi}{2}, \pi]$, we draw it using $\sin(\pi - t) = \sin t$. Finally, in $[\frac{\pi}{2}, \pi]$, we draw it using the fact $\sin(\pi + t) = -\sin t$. The graph is now sketched in Fig. 2.11. The graphs of other trigonometric functions $y = \cos x$, $y = \tan x$, $y = \cot x$, $y = \sec x$, $y = \operatorname{cosec} x$ are also given in Fig. 2.11.

Note: Observe that $\sin\theta = 0$ for $\theta = n\pi$ and $\cos\theta = 0$ for $\theta = (2n+1)\frac{\pi}{2}$, where n is any integer.

Example 2.19

Graph $f(x) = 3\cos 2x$

Let $y = 3\cos 2x$ $\therefore$ $-3 \leq y \leq 3$

Period of $f = \frac{2\pi}{2} = \pi$.

The graph of the curve in the interval $[0, \pi]$ is sketched in Fig. 2.12.

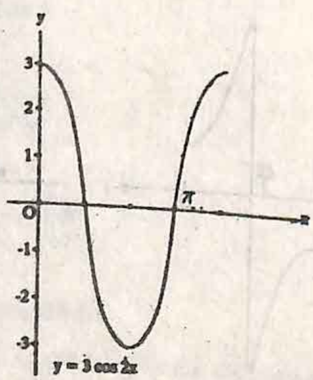


Fig 2.12

Example 2.20

Graph $f(x) = 3\sin(2x - 1)$

Let $y = 3\sin(2x - 1)$

Period of $f = \frac{2\pi}{2} = \pi$

range $-3 \leq y \leq 3$.

Suppose we wish to write

$y = 3\sin(2x - 1) = 3\sin 2t$

or $2x - 1 = 2t$

or $t = x - \frac{1}{2}$

or $x = t + \frac{1}{2}$

Thus if we draw the graph of $3\sin 2t$ and 'shift' it by $\frac{1}{2}$ to the right, we get the required graph. The graph is drawn in Fig. 2.13.

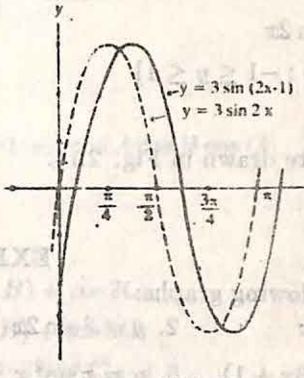


Fig 2.13

Example 2.21

Sketch the graph of $y = \sin x$ and $y = \sin 2x$ on the same axes.

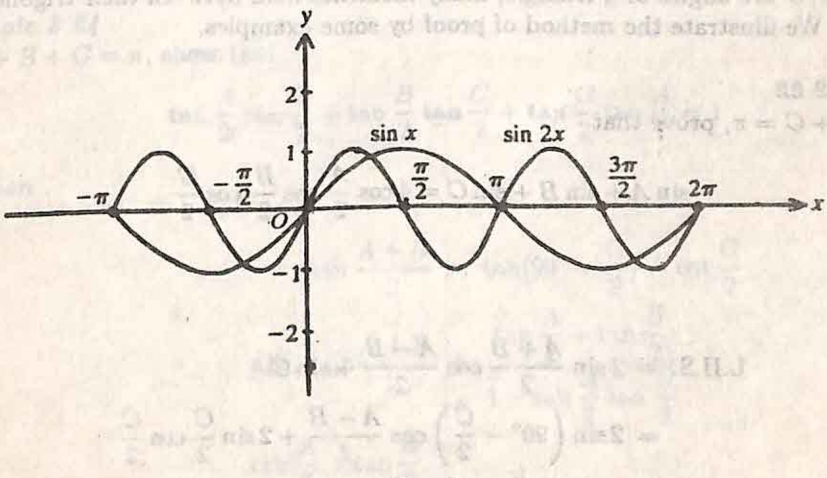


Fig 2.14

Solution

$$
\begin{aligned}
y &= \sin x \\
\text{Range} &= \{y : -1 \le y \le 1\} \\
\text{Period} &= 2\pi \\
y &= \sin 2x \\
\text{Range} &= \{y : -1 \le y \le 1\} \\
\text{Period} &= \pi
\end{aligned}
$$

The graphs are drawn in Fig. 2.14.

EXERCISE 2.5

Sketch the following graphs:

1. $y = \tan 3x$ 2. $y = 3\sin 2x$ 3. $y = \cos(x - \frac{\pi}{4})$
4. $y = 3\sin(3x+1)$ 5. $y = x\sin^2 x$ 6. $y = \cos^2 x$

Sketch the graph of the following pair of equations on the same axes:

7. $y = \cos x, y = \cos \frac{1}{2}x$ 8. $y = \sin x, y = \sin(x + 45°)$
9. $y = \tan x, y = \tan(x - 45°)$ 10. $y = \cos 2x, y = \cos(2x - \pi)$

2.8 Conditional Identities Involving the Angles of a Triangle

When A, B, C are angles of a triangle, many identities hold between their trigonometric functions. We illustrate the method of proof by some examples.

Example 2.22
If $A + B + C = \pi$, prove that

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Solution

$$
\begin{aligned}
\text{L.H.S.} &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin C \\
&= 2\sin\left(90° - \frac{C}{2}\right)\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\left(90° - \frac{C}{2}\right)\right)
\end{aligned}
$$

$$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)$$
$$= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right)$$
$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \text{R.H.S.}$$

Example 2.23
If $A+B+C=\pi$, show that

$$\cos 2A+\cos 2B+\cos 2C = -1-4\cos A\cos B\cos C$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= 2\cos(A+B)\cos(A-B)+\cos 2C \\
&= 2\cos(\pi-C)\cos(A-B)+\cos 2C \\
&= -2\cos C\cos(A-B)+2\cos^2 C-1 \\
&= -1+2\cos C\,\{\cos C-\cos(A-B)\} \\
&= -1+2\cos C\,[\cos\{\pi-(A+B)\}-\cos(A-B)] \\
&= -1-2\cos C\,\{\cos(A+B)+\cos(A-B)\} \\
&= -1-2\cos C\,\{2\cos A\cos B\} \\
&= -1-4\cos A\cos B\cos C = \text{R.H.S.}
\end{aligned}$$

Example 2.24
If $A+B+C=\pi$, show that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

Solution

$$\tan\frac{A+B}{2} = \tan(90^\circ-\frac{C}{2}) = \cot\frac{C}{2}$$

$$\text{Also } \tan\frac{A+B}{2} = \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}$$

$$\text{Hence, } \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}} = \frac{1}{\tan\frac{C}{2}}$$

or $$\tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

or $$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1,$$

which is what we wanted to prove.

EXERCISE 2.6

If $A+B+C=\pi$, show that

1. $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$
2. $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$
3. $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.
4. $\cos^2 A+\cos^2 B-\cos^2 C=1-2\ \sin A\sin B\cos C$
5. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
6. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$
7. $\cot B\cot C+\cot C\cot A+\cot A\cot B=1$
8. $\tan A+\tan B+\tan C=\tan A\tan B\tan C$
9. $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
10. $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C$

2.9 Trigonometric Equations

In solving trigonometric equations, we come across situations of the type $\sin\theta=\sin\alpha$ or $\cos\theta=\cos\alpha$ or $\tan\theta=\tan\alpha$. Before illustrating the technique of solving trigonometric equations by some examples, we find the general value of all angles having a given sine, cosine or tangent.

1. Suppose $\sin\theta=\sin\alpha$
   Then $\sin\theta-\sin\alpha=0$
   or $2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}=0$
   Hence, either $\cos\frac{\theta+\alpha}{2}=0$ or $\sin\frac{\theta-\alpha}{2}=0$.
   Therefore, either

$$\frac{\theta+\alpha}{2} = \text{any odd multiple of } \pi/2$$

or $$\frac{\theta-\alpha}{2} = \text{any multiple of } \pi \text{ (See section 2.7)}$$

Thus either

$$\theta = -\alpha + \text{any odd multiple of } \pi$$

or
$$\theta = \alpha + \text{any even multiple of } \pi$$

Thus the general value of θ such that $\sin\theta = \sin\alpha$ is given by

$$\theta = n\pi + (-1)^n \alpha, \text{ where } n \text{ is an integer.}$$

2. Suppose $\cos\theta = \cos\alpha$
Then $\cos\theta - \cos\alpha = 0$
Hence, $-2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0.$
Therefore, either $\frac{\theta+\alpha}{2}$= any multiple of π.

or
$$\frac{\theta-\alpha}{2} = \text{any multiple of } \pi\,.$$

Hence, the general solution for θ such that $\cos\theta = \cos\alpha$ is given by

$$\theta = 2n\pi \pm \alpha, \text{ where } n \text{ is an integer.}$$

3. Suppose $\tan\theta = \tan\alpha$
Then $\frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$
or $\sin\theta\cos\alpha - \sin\alpha\cos\theta = 0$
or $\sin(\theta - \alpha) = 0$
or $\theta - \alpha = n\pi$.
Hence the general solution for θ satisfying $\tan\theta = \tan\alpha$ is
$\theta = n\pi + \alpha$, where n is an integer.

Example 2.25

Solve the equation

$$\sin\theta + \sin 3\theta + \sin 5\theta = 0$$

Solution

Given $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

$$\begin{aligned}
\text{or} \quad & (\sin\theta + \sin 5\theta) + \sin 3\theta = 0 \\
\text{or} \quad & 2\sin\frac{5\theta+\theta}{2}\cos\frac{5\theta-\theta}{2} + \sin 3\theta = 0 \\
\text{or} \quad & 2\sin 3\theta\cos 2\theta + \sin 3\theta = 0 \\
\text{or} \quad & \sin 3\theta(2\cos 2\theta + 1) = 0 \\
\text{or} \quad & \sin 3\theta = 0 \qquad \text{or, } \cos 2\theta = -\frac{1}{2}
\end{aligned}$$

When $\sin 3\theta = 0$, then $3\theta = n\pi$ or $\theta = \frac{n\pi}{3}$.

When $\cos 2\theta = -\frac{1}{2}$, then $2\theta = 2n\pi \pm \frac{2\pi}{3}$ or $\theta = n\pi \pm \frac{\pi}{3}$.

This yields $\theta = (3n+1)\frac{\pi}{3}$ or $(3n-1)\frac{\pi}{3}$

These solutions are contained in the solution set of $\sin 3\theta = 0$.
Hence the required solution set is given by $\left\{\theta : \theta = \frac{n\pi}{3}, n \text{ an integer}\right\}$

Example 2.26

Solve

$$\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$$

Solution

Divide the equation by 2 to get

$$\frac{\sqrt{3}}{2}\cos\theta = \frac{1}{2}\sin\theta = \frac{1}{\sqrt{2}}$$

$$\text{or } \cos\frac{\pi}{6}\cos\theta + \sin\frac{\pi}{6}\sin\theta = \cos\frac{\pi}{4}$$

$$\text{or } \cos(\frac{\pi}{6} - \theta) = \cos\frac{\pi}{4} \text{ or } \cos(\theta - \frac{\pi}{6}) = \cos\frac{\pi}{4}$$

Hence, the solution is $\theta - \frac{\pi}{6} = 2m\pi \pm \frac{\pi}{4}$

or $\theta = 2m\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$

or $\theta = 2m\pi - \frac{\pi}{4} + \frac{\pi}{6}$ or $2m\pi + \frac{\pi}{4} + \frac{\pi}{6}$

$\therefore \theta = 2m\pi - \frac{5\pi}{12}$

or $\theta = 2m\pi - \frac{\pi}{12}$

Note: The above method can be used to solve equations of the form

$$a\cos\theta + b\sin\theta = c$$

Divide the equation by $\sqrt{a^2+b^2}$ to get

$$\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

Let $\tan\alpha = \frac{b}{a}$. Then

$$\sin\alpha \frac{b}{\sqrt{a^2+b^2}}, \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$$

The equation now becomes

$$\cos\alpha\cos\theta + \sin\alpha\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

Hence, $\cos(\theta-\alpha) = \dfrac{c}{\sqrt{a^2+b^2}}$

This will have solutions if

$\left|\dfrac{c}{\sqrt{a^2+b^2}}\right| \le 1$, because, in this case, we can find $(\theta-\alpha)=\beta$ (say), so that

$$\cos\beta = \frac{c}{\sqrt{a^2+b^2}}.$$

The equation can be easily solved now.

Example 2.27

Solve

$$2\cos^2 t + 3\sin t = 0.$$

Solution

The equation yields

$$2(1-\sin^2 t) + 3\sin t = 0$$

or

$$2\sin^2 t - 3\sin t - 2 = 0$$

or

$$(2\sin t + 1)(\sin t - 2) = 0$$

Hence, either $2\sin t + 1 = 0$ or $\sin t - 2 = 0$. But this last situation cannot occur. Hence, $\sin t = -\frac{1}{2} = \sin\frac{7\pi}{6}$.

Therefore, the solution is $t = n\pi + (-1)^n\frac{7\pi}{6}$.

EXERCISE 2.7

Solve the following equations:

1. $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$
2. $\sin 2\theta + \cos\theta = 0$
3. $\sec^2 2\theta = 1 - \tan 2\theta$
4. $\sin\theta = \tan\theta$
5. $\sin 3\theta + \cos 2\theta = 0$
6. $\sin x + \cos x = \sqrt{2}\cos A$
7. $4\cos\theta - 3\sec\theta = \tan\theta$
8. $\sin m\theta + \sin n\theta = 0$
9. $2\tan\theta - \cot\theta = -1$
10. $\cot^2 x + \dfrac{3}{\sin x} + 3 = 0$

CHAPTER 3

Mathematical Induction

3.1 Introduction

The word 'induction' means the method of inferring a general statement from the validity of particular cases. We must be cautious here that in mathematics this kind of inference is not allowed, even when a huge list of particular cases have been verified. Mathematical induction is a principle by which one can conclude a statement for all positive integers, after proving certain related propositions.

Let us see an example to explain the need for our caution.

We know that the numbers 13, 23, 43, 53, 73, etc. are prime numbers. And the numbers 33, 63, 93, etc. are composite. From these particular cases we formulate a general statement: A number of the form $10n + 3$ is prime if n is not divisible by 3. Is this a true statement?

Even if there are hundreds of particular cases where this is known to be true, we cannot conclude that this general statement is true.

In fact this statement is not true in general when the number 143 is of the form $10n + 3$ with $n = 14$, but it is not a prime.

We say that 143 is a counter example to the statement.

Even when we do not have a counter example, we cannot conclude that a general statement is true simply because it has been found to be true in all its particular cases that have been verified. We can at the best say that it is a reasonable conjecture.

This raises the question: How shall we prove a general statement after that is known to be true in some particular cases? We shall formulate in the next two sections, one such method called the principle of mathematical induction.

3.2 Preparation for Induction

A notation: Consider the statements of the form:

1. n is divisible by 3.
2. The number $10n + 3$ is prime.
3. $2^n > n$.

All these are statements concerning the natural numbers $n = 1, 2, 3, \ldots$. We use the notations $P(n)$ or $P_1(n)$ or $P_2(n)$ etc. to denote such statements. When we give values for $n = 1, 2, \ldots$, we obtain particular statements. If in the statement $P(n)$, we substitute $n = 3$, the particular statement so obtained, is denoted by $P(3)$. Let us see some more examples.

Example 3.1
If $P(n)$ is the statement "$n(n+1)$ is even", then what is $P(4)$?

Solution
$P(4)$ is the statement "4(4+1) is even" i.e., "20 is even".

Example 3.2
If $P(n)$ is the statement "$n^3 + n$ is divisible by 3", is the statement $P(3)$ true? Is the statement $P(4)$ true?

Solution
$P(3)$ is "30 is divisible by 3". It is true.
$P(4)$ is "68 is divisible by 3". It is not true.

Example 3.3
Let $P(n)$ be the statement "$3^n > n$." Is $P(1)$ true?

Solution
$P(1)$ is the statement "$3^1 > 1$", i.e. "$3 > 1$", so $P(1)$ is true.

Example 3.4
Let $P(n)$ be the statement "$3^n > n$". What is $P(n+1)$?

Solution
$P(n+1)$ is the statement "$3^{n+1} > n + 1$".

Example 3.5
Let $P(n)$ be the statement "$3^n > n$." If $P(n)$ is true, prove that $P(n+1)$ is true.

Solution
We are given that $3^n > n$, and we wish to prove that $3^{n+1} > n + 1$.
Now $3^{n+1} = 3.3^n > 3.n$.
But $3n > n + 1$ (because for every natural number n, $2n > 1$). So $3^{n+1} > n + 1$.
This proves that if $P(n)$ is true, then $P(n+1)$ is true.

EXERCISE 3.1

1. If $P(n)$ is the statement "$n(n+1)(n+2)$ is divisible by 12," prove that $P(3)$ and $P(4)$ are true, but that $P(5)$ is not true.
2. If $P(n)$ is the statement. "$n^2 > 100$," prove that whenever $P(r)$ is true, $P(r+1)$ is also true.
3. If $P(n)$ is the statement "$2^n \geq 3n$," and if $P(r)$ is true, prove that $P(r+1)$ is also true.
4. If $P(n)$ is the statement "$2^{3n} - 1$ is an integral multiple of 7," prove that $P(1), P(2)$ and $P(3)$ are true.
5. If $P(n)$ is the statement as in problem 4, and if $P(r)$ is true, prove that $P(r+1)$ is true.
6. Give an example of a statement $P(n)$ such that it is true for all n.
7. Give an example of a statement $P(n)$ such that $P(3)$ is true, but $P(4)$ is not true.

3.3 The Principle of Mathematical Induction

Now we are ready to state the principle of mathematical induction:

Let $P(n)$ be a statement such that

(a) $P(1)$ is true

(b) $P(r+1)$ is true whenever $P(r)$ is true.

Then $P(n)$ is true for all natural numbers n.

We shall illustrate this principle by numerous examples.

Example 3.6

Let $P(n)$ be the statement "$n^2 + n$ is even".

Then (a) $P(1)$ is the statement "2 is even". It is true.

(b) If $P(r)$ is true for some r, then to prove that $P(r+1)$ is true, consider

$$\begin{aligned}(r+1)^2 + (r+1) &= r^2 + 2r + 1 + r + 1 \\ &= r^2 + r + 2(r+1) \\ &= \text{an even number } + 2(r+1) \qquad \text{(because } P(r) \text{ is true)} \\ &= \text{sum of two even numbers} \\ &= \text{an even number}\end{aligned}$$

Thus $P(r+1)$ is proved to be true, assuming that $P(r)$ is true.

Therefore (since we have proved both the steps (a) and (b) required for the principle of induction), it follows by the principle of induction that $P(n)$ is true for all n. Note

that the conclusion is so strong that it contains infinite number of statements one for each n.

Example 3.7
Let $P(n)$ be the statement "$n^3 + n$ is divisible by 3".
Here, $P(1)$ becomes "2 is divisible by 3". This is not true.
Therefore, the principle of induction does not apply.

Example 3.8
Let $P(n)$ be the statement "$n^2 > 100$."
Here, $P(1)$ is not true.
Therefore, the principle of induction does not apply. Note, however, that the second part namely 'if $P(r)$ is true, then $P(r+1)$ is true', can be proved here. Still, because $P(1)$ is not true, we conclude that $P(n)$ fails for some values of n.

Example 3.9
Prove that 7 divides $2^{3n} - 1$ for all positive integers.

Solution
Let $P(n)$ be the statement that 7 divides $2^{3n} - 1$.
Then (a) $P(1)$ is the statement "7 divides $2^3 - 1$". This is seen to be true.
(b) Suppose $P(r)$ is true. Then to prove that $P(r+1)$ is true.

$$
\begin{aligned}
\text{Consider } 2^{3(r+1)} - 1 &= 2^{3r+3} - 1 \\
&= 2^{3r}.2^3 - 1 \\
&= 2^{3r}.8 - 1 \\
&= (2^{3r} - 1)8 + 8 - 1 \\
&= (\text{ a multiple of } 7) + 7 \qquad \text{(because } P(r) \text{ is true)} \\
&= \text{a multiple of } 7.
\end{aligned}
$$

Therefore, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n.

EXERCISE 3.2

Prove the following by the principle of induction:

1. The sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$.
2. $n(n+1)(n+2)$ is divisible by 6, where n is a natural number.
3. $1 + 4 + 7 + \ldots + (3n-2) = \dfrac{n(3n-1)}{2}$.
4. If 3^{2n}, where n is a natural number, is divided by 8, the remainder is always 1.

5. $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$.

6. If x and y are any two distinct integers, then $x^n - y^n$ is an integral multiple of $x - y$.

7. The sum $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3 for any positive integer n.

8. $1^2 + 2^2 + \ldots\ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for every positive integer n.

9. $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

10. If a set has n elements, prove that it has 2^n subsets.

MISCELLANEOUS EXERCISE ON CHAPTER 3

1. Prove by induction that the sum of first n odd natural numbers is n^2.

2. If we take any three consecutive natural numbers, prove that the sum of their cubes is always divisible by 9.

3. Prove by induction the inequality $(1+x)^n \geq 1 + nx$ whenever x is positive and n is a positive integer.

4. If $P(n)$ is the statement $n^2 - n + 41$ is prime, prove that $P(1)$, $P(2)$ and $P(3)$ are true. Prove also that $P(41)$ is not true. How does this not contradict the principle of induction?

5. Prove by induction that $2n + 7 < (n+3)^2$ for all natural numbers n. Using this, prove by induction that $(n+3)^2 < 2^{n+3}$ for all natural numbers n.

6. Prove that for $n \in N$
$10^n + 3.4^{n+2} + 5$ is divisible by 9.

7. Prove that $10^{2n-1} + 1$ is divisible by 11 and for all $n \in N$.

8. Prove that
$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}, n \in N.$

9. Prove that
$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$ for every positive integer n.

CHAPTER 4

Cartesian System of Rectangular Coordinates

4.1 Introduction

The geometry you have been studying thus far in earlier classes is called *Euclidean Geometry*. Its approach, as you would recall, was to start with certain *concepts*, like the concepts of points, lines and planes; attribute certain properties to them which we called *axioms* or *postulates* (suggested by physical experience); and then using the methods of deductive logic to derive a number of *theorems* which formed the main fruit of our mathematical activity and revealed to us the interesting and useful properties of the geometric figures under consideration. This approach was first presented by the Greek mathematician *Euclid*, around 300 BC, in his famous treatise "Elements" comprising thirteen books, and is being followed since then. As you would also recall, this made essentially no use of the process of algebra, and is called the *synthetic* approach to geometry.

This was the only approach to geometry for some two thousand years till the French philosopher and mathematician Rene Descartes (1596-1665) published *La Geometrie* in 1637 wherein he introduced the analytic approach (as against synthetic) by systematically using algebra in his study of geometry. This was achieved by representing points in the plane by ordered pairs of real numbers (called *Cartesian Coordinates* named after Rene Descartes), and representing lines and curves by algebraic equations. This wedding of algebra and geometry is known as *analytic* or *coordinate geometry*, and this is what we propose to study here.

4.2 Cartesian Coordinate System

In this section we shall establish a 1-1 correspondence between points on a straight line and the real numbers, and subsequently a 1-1 correspondence between points in the Euclidean plane and ordered pairs of real numbers. This would make it possible to apply the methods of algebra to study problems in geometry.

We are familiar with the representation of real numbers on a line, which we call the *real line*, or the number line, and denote by R^1 (or R). This was achieved through directed line segments and fixing a unit for length measurement. Fix a point O on the line,

which we shall call the *origin* from where all distances should be measured. This divides the line into two parts, the points on the left and right of the origin O. The distances measured (in terms of the fixed units) in the two parts are taken to be of opposite signs. This gives us the idea of directed line segments where not only length, but directions are also taken into account. If A is any other point on the line, then the line segment OA will be called directed line segment, directed from O to A. Obviously then, as directed line segments, $\overrightarrow{OA} = -\overrightarrow{AO}$. Distances measured to the right are conventionally taken as positive, and those measured to the left as negative. Thus every point P on this line corresponds to a real number x whose magnitude is the length OP measured in the prescribed units, and whose sign is +ve or −ve according as P is to the right or left of the origin O. Conversely, given a real number x we can always find a point P on the line on the right or left of O depending on the sign of x, such that the length OP equals $|x|$ units. This establishes a 1-1 correspondence between the points on the line and real numbers.

We now proceed to define a 1-1 correspondence between the points in the Euclidean plane and the set of all *ordered pairs of real numbers (a, b)*. This can be done by defining what is called a *Cartesian Coordinate System* on the Euclidean plane, which we do as under:

In the Euclidean plane draw a horizontal line $X'OX$, a vertical line $Y'OY$ intersecting at O, the *origin*. We select a convenient unit of length and starting from the origin as zero, mark off a number scale on the horizontal line, positive to the right and negative to the left. We mark off the *same scale* on the vertical line, positive upwards and negative downwards of the origin O.

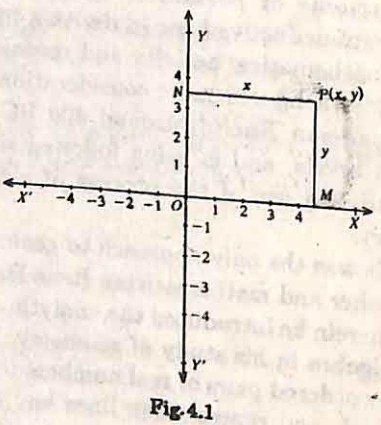


Fig. 4.1

The horizontal line thus marked is called the *x-axis* and the vertical line the *y-axis*, and collectively they are called the *coordinate axes*.

Let P be any point in the plane. Draw perpendiculars from P to the coordinate axes, meeting the x-axis in M and the y-axis in N (Fig. 4.1). Let x be the length of the *directed line segment* OM in the units of the scale chosen. This is called the *x-coordinate* or *abscissa* of P. Similarly, the length of the directed line segment ON in the same scale is called the *y-coordinate* or *ordinate* of P. The position of the point P in the plane with respect to the coordinate axes is represented by the *ordered pair* (x, y) of real numbers, writing the abscissa first in the parenthesis. The pair (x, y) is called the *coordinates* of P, and this system of coordinating an ordered pair (x, y) with every point of the plane is called the (Rectangular) *Cartesian Coordinate System*.

We thus see that to every point P in the Euclidean plane there corresponds a unique

ordered pair (x, y) of real numbers called its Cartesian Coordinates. Conversely, given an ordered pair (x, y) and a cartesian coordinate system, we mark off a directed line segment $OM = x$ on the x-axis and another directed line segment $ON = y$ on y-axis, draw perpendiculars at M and N to x and y-axes respectively, and their point of intersection shall uniquely locate the corresponding point P in the Euclidean plane. This estabilishes a 1-1 correspondence between the set of all ordered pairs (x, y) of real numbers and the points in the Euclidean plane. The set of all ordered pairs (x, y) of real numbers is called Cartesian plane or simply plane and is denoted by R^2.

Finally we observe (Fig. 4.2) that the two axes divide the plane into four regions called the *quadrants*. The ray OX is taken as positive x-axis, OX' as negative x-axis, OY as positive y-axis and OY' as negative y-axis. The quadrants are thus characterised by the following signs of abscissa and ordinate.

I	quardant	$x > 0, y > 0$	or (+,+)
II	quardant	$x < 0, y > 0$	or (–,+)
III	quardant	$x < 0, y < 0$	or (–,–)
IV	quardant	$x > 0, y < 0$	or (+,–)

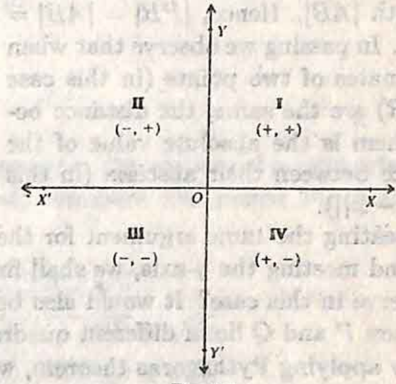


Fig.4.2

Further if the abscissa of a point is zero, it would lie somewhere on the y-axis and if its ordinate is zero it would lie on x-axis. Thus by simply looking at the coordinates of a point we can tell in which quadrant it would lie, e.g. the points (3, 4), (1, -2), (-2, -3) and (-4, 5) lie respectively in I, IV, III and II quadrants.

4.3 Distance Formula

The distance between any two points in the plane is the length of the line segment joining them. Let the coordinates of these two points P and Q be (x_1, y_1) and (x_2, y_2) respectively. We shall sometimes refer to them as points $P(x_1, y_1)$ and $Q(x_2, y_2)$ and obtain, as under, a formula for the distance between them.

Theorem 4.1
The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two points in the plane, and let d be the distance between them (Fig. 4.3). Draw lines parallel to y-axis from the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ which will meet the x-axis in points $A(x_1, 0)$ and $B(x_2, 0)$ respectively. Now draw a line through $P(x_1, y_1)$ parallel to x-axis which will meet the vertical through

Q in $R(x_2, y_1)$. The length of the segment between P and R, which we shall denote by $|PR|$, is equal to $|AB|$.

As in the figure, $|AB| = |OB| - |OA| = (x_2 - x_1)$. If, however, x_2 were to the left of x_1 (i.e. $x_2 < x_1$), this length were $(x_1 - x_2)$. In other case, since the length has got to be positive, we take the absolute value of $(x_2 - x_1)$, viz. $|x_2 - x_1|$ as the length $|AB|$. Hence, $|PR| = |AB| = |x_2 - x_1|$. In passing we observe that when the ordinates of two points (in this case P and R) are the same, the distance between them is the absolute value of the difference between their abscissa (in this case $|x_2 - x_1|$).

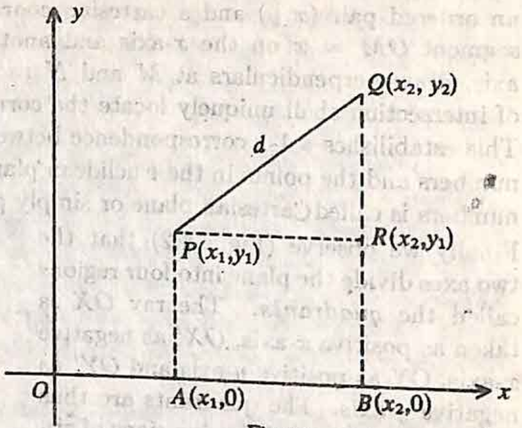


Fig.4.3

Repeating the same argument for the points Q and R, by drawing lines parallel to x-axis and meeting the y-axis, we shall find that the length $|RQ| = |y_2 - y_1|$. (What do you observe in this case? It would also be a good exercise to check the validity of these facts when P and Q lie in different quadrants).

Now applying Pythagoras theorem, we get

$$|PQ|^2 = |PR|^2 + |RQ|^2 \text{ or } d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

which can also be written as

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Since the distance is always positive, taking the positive square root, we get the ***distance formula***

$$d = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This proves the theorem in case the line PQ is parallel to neither x-axis nor y-axis.

If PQ is parallel to x-axis, then obviously $y_1 = y_2$ and $PQ = |x_2 - x_1|$.

Also, $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$. Hence, the theorem is proved if PQ is parallel to x-axis. A similar proof can be given when PQ is parallel to y-axis.

Corollary: The distance of any point $P(x, y)$ from the origin is $\sqrt{x^2 + y^2}$.

Proof: In the above formula, take the point P as (x, y) and Q as $(0, 0)$, i.e. the origin, to get the result.

Example 4.1

What is the distance between the points (4, 5) and (–3, 2)?

Solution

It is immaterial whether we select (4, 5) or (–3, 2) as P, since the choice effects only the sign of $(x_2 - x_1)$ and $(y_2 - y_1)$. If we take (4, 5) as Q, then

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[4-(-3)]^2 + (5-2)^2} \\ &= \sqrt{(4+3)^2 + (3)^2} \\ &= \sqrt{(7)^2 + (3)^2} \\ &= \sqrt{58} \end{aligned}$$

Example 4.2

Prove that the points (4, 4), (3, 5) and (–1,–1) represents the vertices of a right triangle.
Proof: Let the points (4, 4), (3, 5) and (–1, –1) represent the points P, Q and R respectively. Then,

$$\begin{aligned} PQ &= \sqrt{(3-4)^2 + (5-4)^2} = \sqrt{2} \\ QR &= \sqrt{[-1-(3)]^2 + [-1-(5)]^2} = \sqrt{52} \\ \text{and} \quad PR &= \sqrt{[-1-(4)]^2 + [-1-(4)]^2} = \sqrt{50} \end{aligned}$$

Since $QR^2 = RP^2 + PQ^2$, it follows from the converse of the Pythagoras Theorem that the triangle is a right triangle, with right angle at P.

EXERCISE 4.1

1. Find the distance between each of the following pairs of points:

   (i) $(-1,-4), (3,5)$

   (ii) $(a\cos\alpha, a\sin\alpha), (a\cos\beta, a\sin\beta)$

2. By using the distance formula, prove that each of the following sets of points are the vertices of a right triangle:

   (i) $(6,2), (3,-1), (-2,4)$

   (ii) $(-2,2), (8,-2), (-4,-3)$

3. Show that each of the triangles whose vertices are given below are isosceles:

   (i) $(8,2), (5,-3), (0,0)$

   (ii) $(0,6), (-5,3), (3,1)$

4. Find the value of x such that $PQ = QR$, where P, Q and R are (6, –1), (1, 3) and (x, 8), respectively.
5. What point on the x-axis is equidistant from (7, 6) and $(-3, 4)$?
6. Give the relation that must exist between x and y so that (x, y) is equidistant from $(6, -1)$ and (2, 3).
7. Show that the quadrilateral with vertices $(3,2), (0,5), (-3,2), (0,-1)$ is a square.

4.4 Area of a Triangle

We now proceed to find the area of a triangle ABC, the coordinates of whose vertices are given to be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Draw perpendiculars from A, B and C to x-axis meeting it in L, M and N respectively. As we have seen earlier, $|LM|$ or simply $ML = |x_1 - x_2| = x_1 - x_2$.

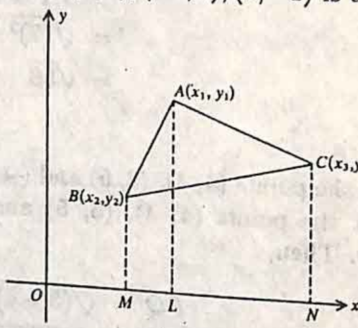


Fig.4.4

Similarly, $LN = (x_3 - x_1)$ and $MN = x_3 - x_2$. Now, the area of $\triangle ABC$ = area of trapezium $BMLA +$ area of trapezium $ALNC -$ area of trapezium $BMNC$ =

$$\frac{1}{2}(MB + AL).ML + \frac{1}{2}(AL + CN).LN - \frac{1}{2}(BM + CN).MN$$
$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$
$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)].$$

The expression

$$\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

is sometimes written in short as

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Such an expression is called a *determinant*.
With this notation, the area of the triangle with vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note

1. Sometimes the above expression for area may turn out to be negative. But we take the absolute value of the expression as the area.
2. The above proof uses the fact that all vertices of the triangle are on one side of the y-axis; it can be shown that even if some vertex or vertices are on the other side of the y-axis the same expression will give us the area of the triangle.
3. We have given the above proof by drawing perpendiculars from vertices on the x-axis. A proof can also be given by drawing perpendiculars on the y-axis.

Condition of Collinearity of Three Points

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, i.e. lie on the same straight line if and only if the area of the ΔABC is zero. Hence, three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example 4.3
Find the area of a triangle whose vertices are (4, 4), (3, –2), and (–3, 16).

Solution
Using the above formula, required area is

$$\frac{1}{2}\begin{vmatrix} 4 & 4 & 1 \\ 3 & -2 & 1 \\ -3 & 16 & 1 \end{vmatrix} = \frac{1}{2}[4(-2-16) - 4(3-(-3)) + 1(48-6)] = -27$$

Rejecting the negative sign, area of triangle = 27 sq. units.
Note: If we actually plot the vertices and take them in anti-clockwise direction, the order would be (4, 4), (–3, 16) and (3, –2) and then the value of the determinant would be +27.

Example 4.4
Show that the three points (–1, –1), (2, 3) and (8, 11) lie on a line.
Proof: We have

$$\frac{1}{2}\begin{vmatrix} -1 & -1 & 1 \\ 2 & 3 & 1 \\ 8 & 11 & 1 \end{vmatrix} = \frac{1}{2}[-1(3-11) + 1(2-8) + 1(22-24)] = 0$$

Hence the result.

EXERCISE 4.2

1. Find the area of the triangle with vertices at the points given in each of the problems (a) to (d).
   (a) (0, 0), (1, 0), (1, 1)
   (b) (–2, 1), (2, –3), (4, 4)
   (c) (3, 8), (–4, 2), (5, –1)
   (d) (2, 7), (3, –1), (–5, 6)
2. Show that each of the following triple of points are collinear:
   (a) (2, 4), (0, 1), (4, 7)
   (b) (–2, 5), (2, –3), (0, 1)
   (c) (–5, 7), (–4, 5), (1, –5)
3. For what value of x will the points $(x, -1)$, (2, 1) and (4, 5) lie on a line?
4. A and B are two points (3, 4) and (5, –2). Find the coordinates of a point P such that $PA = PB$ and the area of the triangle $PAB = 10$.
   (*Hint:* We can take the area as 10 or –10, hence two different points.)
5. Find the condition that the point (x, y) may lie on the line joining (3, 4) and (–5, –6).

4.5 Section Formula

Theorem 4.2

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$. The coordinates of the point P on AB which divides the line segment AB in the ratio $l : m$ (internally) are given by

$$x = \frac{mx_1 + lx_2}{l+m}, y = \frac{my_1 + ly_2}{l+m}$$

Proof: Draw lines parallel to y-axis from A, B and P meeting x-axis in C, D, and Q respectively. Draw lines parallel to x-axis from A and P meeting PQ and BD in E and R respectively. This being given that $\frac{AP}{PB} = \frac{l}{m}$. It is easily seen that the two right angled triangles APE and PBR are similar, and hence,

$$\frac{AP}{PB} = \frac{AE}{PR} = \frac{PE}{BR} = \frac{\ell}{m}$$

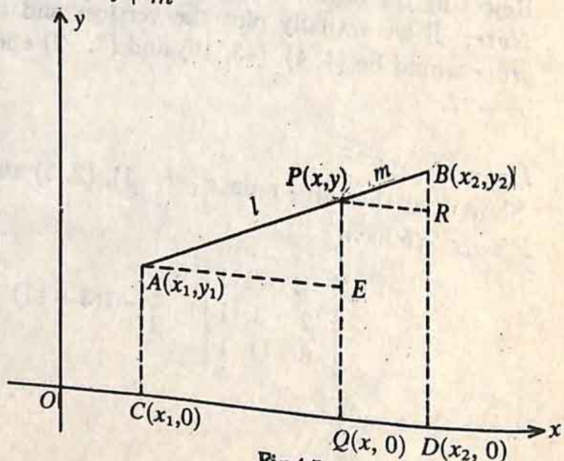


Fig.4.5

Now $AE = CQ = |OQ - OC| = |x - x_1| = x - x_1$
and $PR = QD = |OD - OQ| = |x_2 - x| = x_2 - x$

$$\text{Using } \frac{AP}{PB} = \frac{AE}{PR} = \frac{PE}{BR} = \frac{\ell}{m}$$

$$\frac{\ell}{m} = \frac{AE}{PR} = \frac{x - x_1}{x_2 - x} \text{ or } l(x_2 - x) = m(x - x_1)$$

$$\text{i.e. } x = \frac{mx_1 + lx_2}{l + m}$$

Again, $PE = |PQ - QE| = |PQ - AC| = |y - y_1| = y - y_1$
and $BR = |BD - RD| = |BD - PQ| = |y_2 - y| = y_2 - y$
Using $\frac{\ell}{m} = \frac{PE}{BR} = \frac{y - y_1}{y_2 - y}$, we have $l(y_2 - y) = m(y - y_1)$
i.e. $y = \frac{my_1 + ly_2}{l + m}$

Note: To remember the formula it is helpful to note that l is multiplied by the coordinate 'away from it', and similarly is m, and the sum then divided by $l + m$. This is diagrammatically shown in Fig. 4.6 as aid to memory.

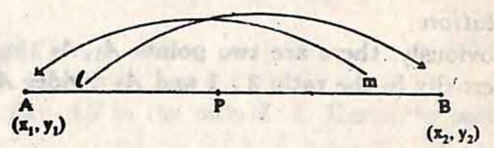


Fig.4.6

External Division

If the line AB is divided externally by a point P in the ration $l : m$ as shown in Fig. 4.7, then it is easy to see that $AP = l$ and $BP = m$, for a suitably chosen unit where P lies on AB produced.
Thus the point $B(x_2, y_2)$ divides the line AP *internally* in the ratio $(l - m) : m$. Our formula for internal division therefore implies that

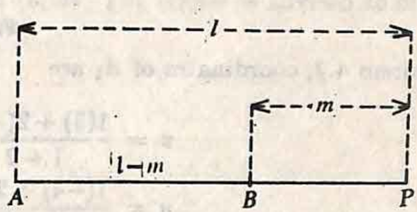


Fig.4.7

$$x_2 = \frac{(l - m)x + mx_1}{(l - m) + m}, y_2 = \frac{(l - m)y + my_1}{(l - m) + m},$$

so that

$$lx_2 = (l - m)x + mx_1 \text{ and } ly_2 = (l - m)y + my_1$$

giving

$$x = \frac{lx_2 - mx_1}{l - m}, y = \frac{ly_2 - my_1}{l - m}$$

Note that this is the same formula as for the internal division except that m is replaced by $-m$. If P divides AB externally in the ratio $l:m$ and $l<m$, then the coordinates of P will be given by

$$x=\frac{-lx_2+mx_1}{-l+m},\; y=\frac{-ly_2+my_1}{-l+m}$$

Mid-Point Formula

To find the coordinates of the ***mid-point*** of a line segment with end points $A(x_1,y_1)$ and $B(x_2,y_2)$, we put $l=m$ in the formula of theorem 4.2 and obtain

$$x=\frac{x_1+x_2}{2},\; y=\frac{y_1+y_2}{2}$$

Example 4.5

Find a point on the line through $A(5,-4)$ and $B(-3,2)$, that is, twice as far from A as from B.

Solution

Obviously, there are two points A_1, A_2 that satisfy this requirement. A_1 divides AB internally in the ratio $2:1$ and A_2 divides AB externally in the same ratio.

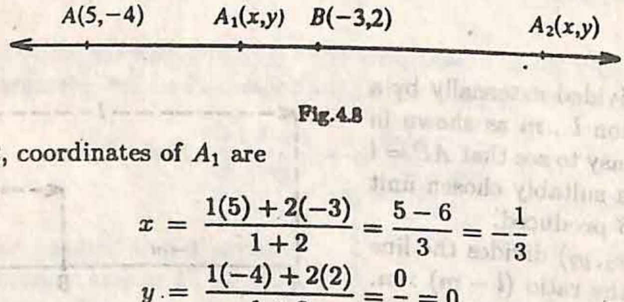


Fig.4.8

By theorem 4.2, coordinates of A_1 are

$$x=\frac{1(5)+2(-3)}{1+2}=\frac{5-6}{3}=-\frac{1}{3}$$
$$y=\frac{1(-4)+2(2)}{1+2}=\frac{0}{3}=0$$

So coordinates of A_1 are $(-\frac{1}{3},0)$.

For A_2, we have

$$x=\frac{-1(5)+2(-3)}{-1+2}=\frac{-5-6}{1}=-11, \qquad y=\frac{-1(-4)+2(2)}{-1+2}=\frac{4+4}{1}=8$$

Therefore, A_2 has coordinates $(-11,8)$.

Example 4.6
Find the centroid of the triangle whose vertices are $A(-1,0), B(5,-2)$ and $C(8,2)$.

Solution
Centroid, the point where the medians of a triangle intersect, divides each median in the ratio 2:1. Coordinates of the mid-point of BC are $(\frac{5+8}{2}, \frac{-2+2}{2})$ i.e. $(\frac{13}{2}, 0)$

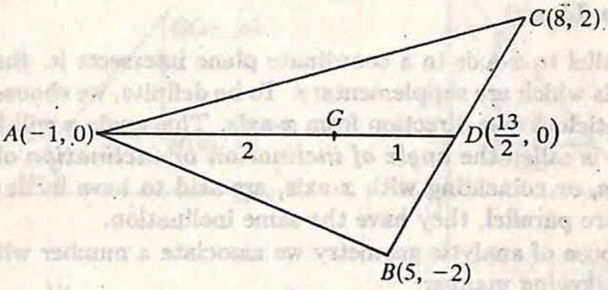


Fig 4.9

If G is the centroid, it must divide the median AD in the ratio $2:1$. Hence, by section formula, its coordinates are

$$x = \frac{2 \times \left(\frac{13}{2}\right) + 1(-1)}{2+1} = 4, \quad y = \frac{2 \times 0 + 1 \times 0}{2+1} = 0$$

Hence, the coordinates of the centroid G are $(4, 0)$. The reader is advised to check the result corresponding to other two medians.

EXERCISE 4.3

1. Find the coordinates of the points which divide internally and externally the line joining $(1,-3)$ and $(-3,9)$ in the ratio 1:3.
2. Prove that the points $(-2,-1), (1,0), (4,3)$ and $(1,2)$ are the vertices of a parallelogram.
(*Hint:* Diagonals of a parallelogram bisect each other)
3. Find the centroid of a triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3).

4. Find the third vertex of a triangle if two of its vertices are at $(-1,4)$ and $(5,2)$ and the centroid at $(0,-3)$.

5. In what ratio does the point $(\frac{1}{2},6)$ divide the line segment joining the points $(3,5)$ and $(-7,9)$?

6. Find the ratio in which the line segment joining $(2,-3)$ and $(5,6)$ is divided by x-axis.

4.6 Slope of a Line

A line l not parallel to x-axis in a coordinate plane intersects it. Such a line forms two angles with x-axis which are supplementary. To be definite, we choose that angle x which is made going anticlockwise direction from x-axis. This angle x will have values between $0°$ and $180°$, and is called the *angle of inclination or inclination* of the line l. All lines parallel to x-axis, or coinciding with x-axis, are said to have inclination $0°$. Note that when two lines are parallel, they have the same inclination.

For the purpose of analytic geometry we associate a number with the inclination of the line in the following manner:

Definition 4.1

The slope m of a line having inclination α, and not perpendicular to x-axis, is defined to be $\tan\alpha$.

The slope of a line perpendicular to x-axis is not defined as the value of $\tan\alpha$ at $\alpha = 90°$ is undefined.

Note that the slope m is independent of the sense of line segment. As shown in Fig. 4.10 slope of $AB = \tan\alpha = \tan(\pi+\alpha)$ = slope of BA and hence we do not take into consideration the direction of a line segment, while talking of its slope.

We know that a line is fully determined when we are given two points on it. Hence, we proceed to find a formula for the slope of a line in terms of the coordinates of two points given on it.

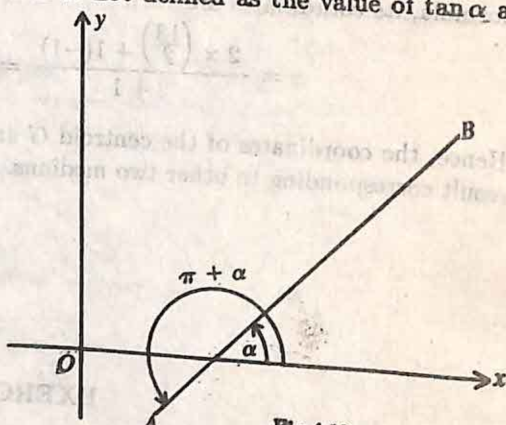


Fig.4.10

Theorem 4.3

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line l, then the slope m of the line l is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1},\ x_1 \neq x_2$$

If $x_1 = x_2$, then m is not defined. In that case the line is perpendicular to x-axis.

Proof: Let α be the inclination of the line l. We shall consider two different cases when α is acute or obtuse, as shown in Fig. 4.11(i) and (ii). Draw horizontal and verticle lines

(i.e. parallel to x-axis and y-axis respectively) through P and Q respectively intersecting at M, whose coordinates are shown in the figure.

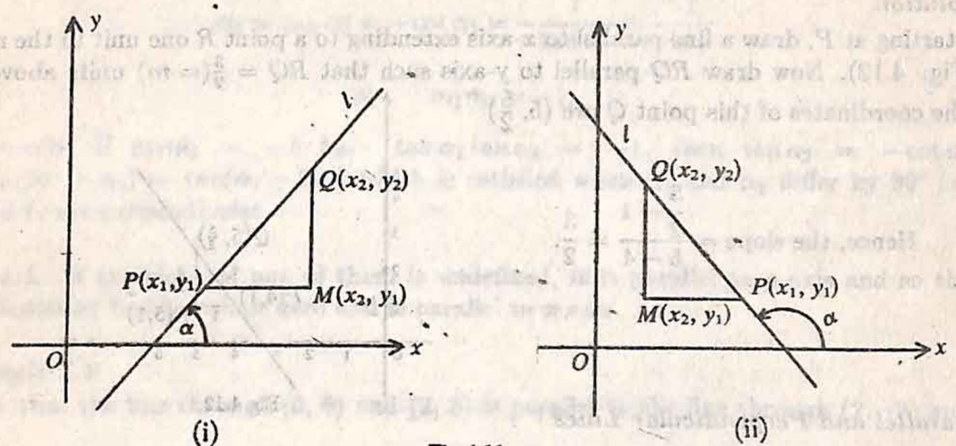


Fig.4.11

In Fig. 4.11(i) the inclination α is equal to $\angle MPQ$, hence $m = \tan\alpha = \tan\angle MPQ = \dfrac{MQ}{PM} = \dfrac{y_2 - y_1}{x_2 - x_1}$

In Fig. 4.11(ii) the inclination α and $\angle MPQ$ are supplementary, hence

$$m = \tan\alpha = \tan(180^\circ - \angle MPQ) = -\tan\angle MPQ = -\frac{MQ}{MP} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

Consequently, we see that in all cases the slope m of the line through points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The difficulty arises when $x_1 = x_2$, i.e. when m is not defined. This is the case when the line l is parallel to y-axis or perpendicular to x-axis.

Thus given any line not perpendicular to x-axis, we can always find its slope by locating two points on it, and also given any point we can draw a line of the given slope through it.

Example 4.7

Find the slope of a line l determined by the points $P(4, 6)$ and $Q(2, 12)$.

Solution

Here $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{12 - 6}{2 - 4} = -3$

Obviously l makes an obtuse angle with x-axis.

Example 4.8

Through $P(4,1)$ construct a line which has slope equal to $\frac{3}{2}$.

Solution

Starting at P, draw a line parallel to x-axis extending to a point R one unit to the right (Fig. 4.12). Now draw RQ parallel to y-axis such that $RQ = \frac{3}{2}(= m)$ units above R. The coordinates of this point Q are $(5, \frac{5}{2})$

$$\text{Hence, the slope} = \frac{\frac{5}{2} - 1}{5 - 4} = \frac{3}{2}.$$

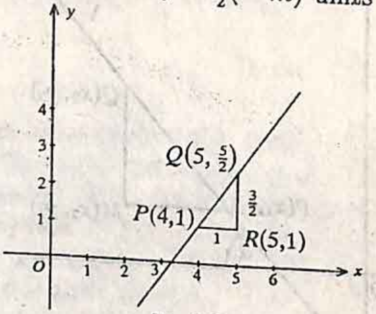


Fig.4.12

Parallel and Perpendicular Lines

If two lines are parallel (neither of which is parallel to y-axis), then their inclinations are same and hence their slopes are also same. Conversely, if the slope m of two lines is the same, then by the property of tangent function there exists a unique angle x between 0° and 180° such that $\tan x = m$, and hence their inclinations are same which means they are parallel. Thus we arrive at the following result:

Theorem 4.4

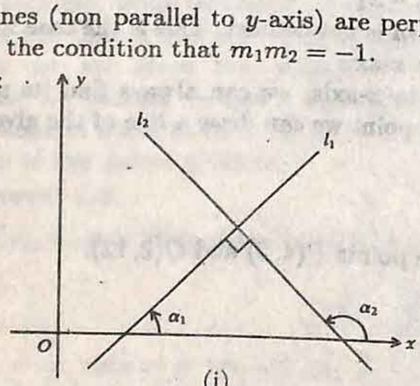
Two lines (not parallel to y-axis) are parallel if and only if their slopes are equal.

The question arises: Can we characterise perpendicularity also in terms of slope? The answer is given by the following theorem:

Theorem 4.5

Two lines (non parallel to y-axis) are perpendicular, if and only if their slopes m_1, m_2 satisfy the condition that $m_1 m_2 = -1$.

Proof:



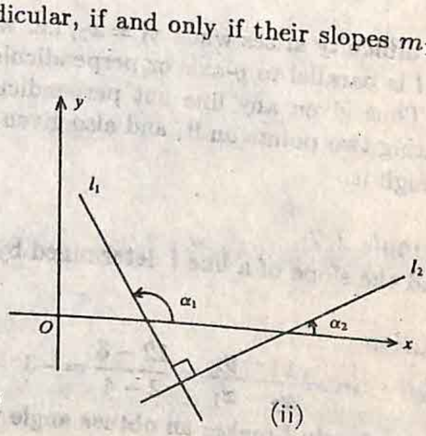


Fig.4.13

As in Fig. 4.13 (i) and (ii), if the lines ℓ_1 and ℓ_2 are perpendicular, then α_1 and α_2 differ by $90°$, i.e. $\alpha_2 = \alpha_1 \pm 90°$, hence $\tan\alpha_2 = \tan(\alpha_1 \pm 90°)$. In either case we get

$$m_2 = \tan\alpha_2 = -\cot\alpha_1 = -\frac{1}{\tan\alpha_1} = -\frac{1}{m_1}$$

$$\text{or} \qquad m_1 m_2 = -1$$

Conversely, if $m_1 m_2 = -1$ i.e. $\tan\alpha_1 \tan\alpha_2 = -1$, then $\tan\alpha_2 = -\cot\alpha_1 = \tan(90° + \alpha_1)$ or $\tan(\alpha_1 - 90°)$ which is satisfied when α_1 and α_2 differ by $90°$ i.e. ℓ_1 and ℓ_2 are perpendicular.

Remark: If the slope of one of them is undefined, it is parallel to y-axis and so the perpendicular line has slope zero and is parallel to x-axis.

Example 4.9

Show that the line through (0, 0) and (2, 3) is parallel to the line through (2, −2) and (6, 4).

Proof:

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-0}{2-0} = \frac{3}{2}$$

$$m_2 = \frac{4-(-2)}{6-2} = \frac{6}{4} = \frac{3}{2}$$

Since the slopes are equal, the lines are parallel.

Example 4.10

Prove that the line through (−2,6) and (4,8) is perpendicular to the line through (8,12) and (4,24).

Proof:

$$m_1 = \frac{8-6}{4-(-2)} = \frac{1}{3} \quad \text{and} \quad m_2 = \frac{24-12}{4-8} = -3.$$

Hence, $m_1 m_2 = -1$ and the lines are perpendicular.

EXERCISE 4.4

1. What can be said regarding a line if its slope is
   (a) positive, (b) zero, (c) negative?
2. What is the slope of a line whose inclination is
   (a) 0°, (b) 45° (c) 90° (d) 150°

3. Find the slope of the line through the points
   (a) (1, 2), (4, 2) (b) (0, –4), (–6, 2) (c) (4, –6), (–2, –5)
4. Show that the line joining (2, –3) and (–5, 1) is
   (a) parallel to the line joining (7, –1) and (0, 3)
   (b) perpendicular to the line joining (4, 5) and (0, –2)
5. State whether the two lines in each of the following are parallel, perpendicular or neither.
   (a) Through (5, 6) and (2, 3); through (9, –2) and (6, –5)
   (b) Through (8, 2) and (–5, 3); through (16, 6) and (3, 15)
   (c) Through (2, –5) and (–2, 5); through (6, 3) and (1, 1)
   (d) Through (9, 5) and (–1, 1); through (8, –3) and (3, –5)
6. Determine x so that 2 is the slope of the line through (2, 5) and $(x, 3)$.
7. What is the value of y so that the line through $(3, y)$ and (2, 7) is parallel to the line through (–1, 4) and (0, 6)?
8. Without using Pythagoras Theorem, show that (4, 4), (3, 5) and (–1, –1) are the vertices of a right triangle.
9. A quadrilateral has the vertices at the points (–4, 2), (2, 6), (8, 5) and (9, –7). Show that the mid-points of the sides of this quadrilateral are the vertices of a parallelogram.

4.7 Sets of Points and Equations

Let us consider the following:
Consider a circle of radius a whose centre is at the origin. This circle is the set of all points in the plane whose distance from the origin is a.
Let $P(x, y)$ be any point on this circle. Then as its distance from origin (0, 0) is a, we have

$$\sqrt{x^2 + y^2} = a$$

and so, for every point $P(x, y)$ on the circle, we get $x^2 + y^2 = a^2$.

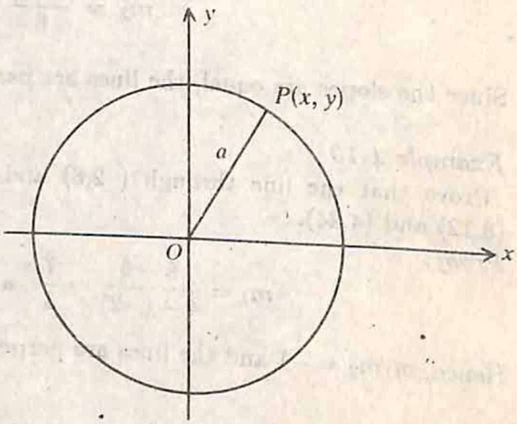


Fig.4.14

Conversely, if (x, y) is any point in the plane satisfying the equation $x^2+y^2 = a^2$, then this shows that the distance of the point (x, y) from the origin is a, so that the point is on the circle. Thus the circle with centre as origin and radius a is the set of all points (x, y) such that $x^2 + y^2 = a^2$.

We say that $x^2 + y^2 = a^2$ is the equation of this circle.

Again, consider the line l parallel to x-axis and lying 4 units above the x-axis.
It is obvious that if $P(x, y)$ is any point on this line, then $y = 4$. Conversely any point (x, y) for which $y = 4$ will lie on this line l. Hence

$$l = \{(x, y) | y = 4\}.$$

So the equation of the line l is $y = 4$.
These examples indicate that if we take a set of points in the plane such as a line or circle, etc. we can find an equation in x and y (or only x or only y) such that a point (x, y) belongs to this set if and only if the coordinates (x, y) satisfy the equation.
In general, different sets of points will have different equations.

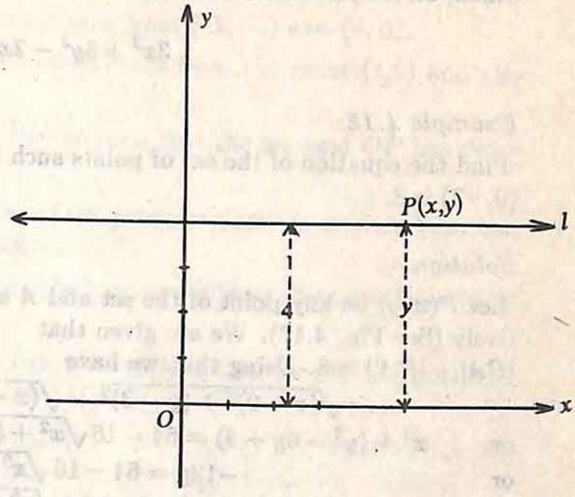


Fig.4.15

Example 4.11

Find the equation of the set of all points which are twice as far from (3, 2) as from (1,1).

Solution

Let A be the point (3, 2) and B the point (1, 1).
Suppose

$$S = \{P | PA = 2PB\}$$

Let $P \in S$ and P have coordinates (x, y).
As then

$$PA^2 = (x-3)^2 + (y-2)^2$$
$$PB^2 = (x-1)^2 + (y-1)^2$$

Now $\qquad PA = 2PB$

So $\qquad PA^2 = 4PB^2$

$$(x-3)^2+(y-2)^2 = 4\{(x-1)^2 + (y-1)^2\}$$

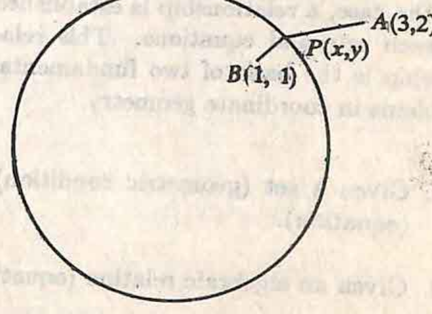


Fig.4.16

which, on simplification becomes

$$3x^2 + 3y^2 - 2x - 4y - 5 = 0$$

Example 4.12

Find the equation of the set of points such that the sum of its distances from (0,3) and (0, –3) is 8.

Solution

Let $P(x, y)$ be any point of the set and A and A' be the points (0,3) and (0,–3) respectively (See Fig. 4.17). We are given that $|PA| + |PA'| = 8$. Using this, we have

$$\sqrt{(x-0)^2 + (y-3)^2} + \sqrt{(x-0)^2 + (y+3)^2} = 8$$

or $$x^2 + (y^2 - 6y + 9) = 64 - 16\sqrt{x^2 + (y+3)^2} + x^2 + y^2 + 6y + 9$$

or $$-12y = 64 - 16\sqrt{x^2 + (y+3)^2}$$

or $$12y + 64 = 16\sqrt{x^2 + (y+3)^2}$$

or $$3y + 16 = 4\sqrt{x^2 + (y+3)^2}$$

or $$9y^2 + 96y + 256 = 16(x^2 + y^2 + 6y + 9)$$

or $$112 = 16x^2 + 7y^2.$$

Hence, $$\frac{x^2}{7} + \frac{y^2}{16} = 1$$

Therefore, the required equation is $\frac{x^2}{7} + \frac{y^2}{16} = 1$.

From the above discussion it is clear that if a coordinate system is defined, the condition determining a particular set may be expressible as an equation or a set of equations involving the coordinates x and y of a point. When it is possible, as is often the case, a relationship is established between sets and equations. This relationship is the basis of two fundamental problems in coordinate geometry.

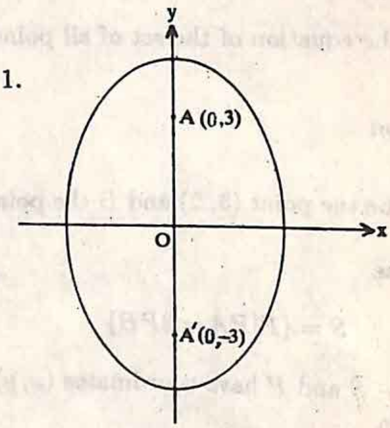


Fig.4.17

1. Given a set (geometric condition), to find the corresponding algebraic relation (equation).

2. Given an algebraic relation (equation), to find the corresponding set.

In the articles that follow, we shall consider these two problems with respect to the sets — the straight lines and circles.

EXERCISE 4.5

1. Find the equation of the set of points equidistant from $(-1, -1)$ and $(4, 2)$.
2. Find the equation of the set of all points equidistant from the point $(4,2)$ and the x-axis.
3. Find the equation of the set of all points $P(x, y)$ such that the segment OP has slope 3, where O is the origin.
4. Find the equation of the set of points for which every ordinate is greater than the corresponding abscissa by a given distance.
5. Find the equation of the set of points such that the sum of their distances from $(0,2)$ and $(0,-2)$ is 6.
6. Find the equation of the set of all points $P(x, y)$ such that the line OP is coincident with the line joining P and the point $(3, 2)$.

CHAPTER 5

Straight Line

5.1 To Find the Equation of a Straight Line Parallel to an Axis

We know that on x-axis, the y-coordinates of all points are zero. We say that the equation of x-axis is $y = 0$. Similarly, the equation of y-axis is $x = 0$. Now the equation of a straight line parallel to x-axis is $y = b$, where b is the ordinate of any point on the line. Similarly, $x = a$ is the equation of a straight line parallel to y-axis, where a is the abscissa of any point on the line.

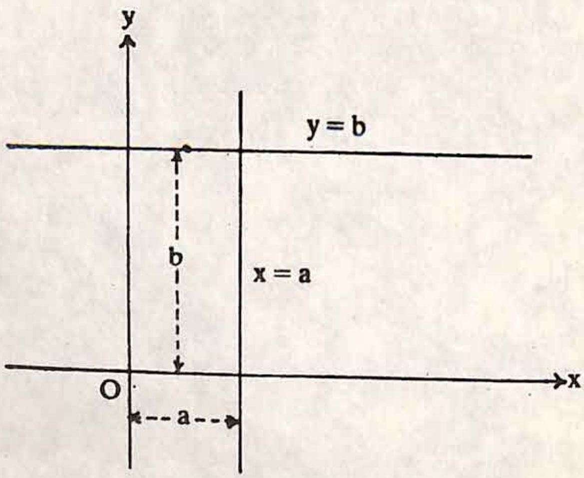


Fig.5.1

5.2 The Point-Slope Form

Now we are in a position to obtain the equation of a line determined by a given set of conditions.

Let $P_1(x_1, y_1)$ be a fixed point and m be a given slope. If any other point on the line is $P(x, y)$, then $\frac{y - y_1}{x - x_1}$ is the slope of the line through P_1 and P. But this is given as m. Hence,

$$\frac{y - y_1}{x - x_1} = m$$

$$\text{or} \quad y - y_1 = m(x - x_1) \tag{5.1}$$

Conversely, if a point $Q(x, y)$ in the plane satisfies (5.1), then as $\frac{y - y_1}{x - x_1}$ is the slope of QP_1, (5.1) expresses the fact that the slope of QP_1 is m. Thus Q is on a line through P_1 with slope m.

Thus, if l is the line through P_1 with slope m, then we have shown that

$$l = \{(x, y) | y - y_1 = m(x - x_1)\}$$

Hence the equation (5.1) is the equation of the line through the point (x_1, y_1) and slope m. This form of the equation of a line is called the point-slope form.

Note that the slope m is undefined for the lines parallel to y-axis. Hence, the point-slope form of the equation will not be applicable to the equation of a line through $P_1(x_1, y_1)$ parallel to the y-axis. However, this presents no difficulty, since for any such line the x-coordinate of any point on it is x_1, the equation of such a line is $x = x_1$.

Example 5.1

Determine the equation of the line passing through the point (–1, –2) and with slope $\frac{4}{7}$.

Solution

Putting the values $x_1 = -1, y_1 = -2$ and $m = \frac{4}{7}$ in the point-slope form of the equation, we get

$$y - (-2) = \frac{4}{7}[x - (-1)]$$

or $$y + 2 = \frac{4}{7}(x + 1)$$

or $$7y + 14 = 4x + 4$$

Therefore, equation of the line is $y = \frac{4}{7}x - \frac{10}{7}$

Example 5.2

Determine the equation of the line through the point (3, –4) and parallel to the x-axis.

Solution

A line parallel to the x-axis has the slope zero. Therefore, point-slope form of the equation gives

$$y - (-4) = 0(x - 3)$$

$$\text{or} \quad y + 4 = 0$$

Another way to approach this problem is to note that every point on the line must have the same ordinate. Since one point has ordinate -4, we must have $y = -4$ for all points.

5.3 Two-Point Form

Let $Q_1(x_1, y_1)$ and $Q_2(x_2, y_2)$ be two points and let l be the line through these two points. If $x_1 = x_2$ then Q_1Q_2 is parallel to x-axis and the equation of l is $x = x_1$.

If $x_1 \neq x_2$, let $P(x, y) \in l$. Then as PQ_1 and Q_2Q_1 are the same lines they have the same slope. So

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

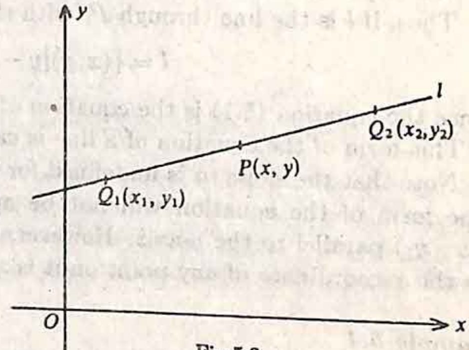


Fig.5.2

Conversely, if a point $Q(x, y)$ satisfies $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$, then this last equation indicates that the slope of QQ_1 = slope of Q_2Q_1. So the lines QQ_1 and Q_1Q_2 are either the same or they are parallel. But these lines already have a common point Q_1, so they are the same lines. Thus Q is on Q_1Q_2, i.e. $Q \in l$. Hence

$$l = \left\{ (x, y) \,\middle|\, \frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \right\}.$$

Therefore equation of the line through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$\text{or} \qquad \frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2} \tag{5.2}$$

5.4 Intercept Form

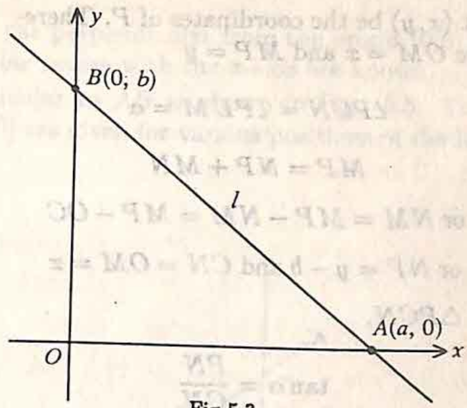


Fig.5.3

Suppose a line l is parallel to neither x-axis nor y-axis. Then l intersects x-axis at some point $A(a, 0)$ and it intersects the y-axis at some point $B(0, b)$. We say that a and b are respectively x-intercept and y-intercept of l. Since l passes through the points $(a, 0)$ and $(0, b)$, we see that the two point form tells us that the equation of l is

$$\frac{y-b}{b-0} = \frac{x-0}{0-a}, \text{ i.e. } \frac{y}{b} - 1 = -\frac{x}{a}$$

or
$$\frac{x}{a} + \frac{y}{b} = 1.$$

Thus the equation of a line whose x and y-intercepts are a and b is

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{5.3}$$

5.5 Slope-Intercept Form

If a line is not parallel to y-axis, it may be determined by its y-intercept and slope m. If a line has the y-intercept b, it passes through the point $(0, b)$. Hence, we may use the point-slope form to obtain the equation of the line in terms of these quantities. Substituting $(0, b)$ for (x_1, y_1) in the point-slope form, we get

$$y - b = m(x - 0)$$

or
$$y = mx + b$$

This is the equation of the line with slope m and y-intercept b. This form of the equation of the straight line is very useful.

Alternative Method

Let the given intercept of a line l with y-axis be c and inclined at an angle α with x-axis.

Let C be the point on y-axis such that $OC = b$.

Through C draw a straight line inclined at an angle $\alpha[= \tan^{-1}(m)]$ to x-axis so that $\tan\alpha = m$. Let P be any point on the line. Draw PM perpendicular to x-axis to meet a line through C parallel to x-axis in N.

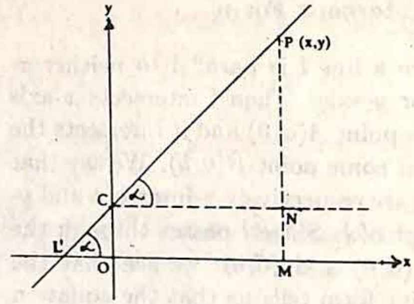


Fig.5.4

Let (x, y) be the coordinates of P. Therefore $OM = x$ and $MP = y$

$$\angle PCN = \angle PL'M = \alpha$$

$$MP = NP + MN$$

$$\text{or } NM = MP - NM = MP - OC$$

$$\text{or } NP = y - b \text{ and } CN = OM = x$$

In $\triangle PCN$,

$$\tan \alpha = \frac{PN}{CN}$$

$$\text{or } m = \frac{y-b}{x}$$

$$\text{or} \quad y = mx + b \tag{5.4}$$

This is the equation of the line with slope m and y-intercept b.

Example 5.3

What is the equation of a line with slope 3 and y-intercept 2?

Solution

On substituting $m = 3$ and $b = 2$, in the slope-intercept form of the equation, we get $y = 3x + 2$.
This is the desired equation.

Example 5.4

Determine the slope and the y-intercept of the line whose equation is $5x + 6y = 7$.

Solution

Expressing y in terms of x we have

$$y = -\frac{5}{6}x + \frac{7}{6}$$

Comparing this equation with the slope-intercept form, we see that $m = -\frac{5}{6}$ and $b = \frac{7}{6}$.
Therefore, slope of the line is $-\frac{5}{6}$ and its y-intercept is $\frac{7}{6}$.

5.6 Normal Form

A straight line is determined if the length of the perpendicular from the origin (0,0) to the line, and the angle which this perpendicular makes with the x-axis are known.

Let AB be the line. Draw OP perpendicular to AB as shown in Fig. 5.5. Four different figures [See Fig. 5.5 (i), (ii), (iii), (iv)] are given for various positions of the line AB. Consider the first Fig. 5.5 (i).

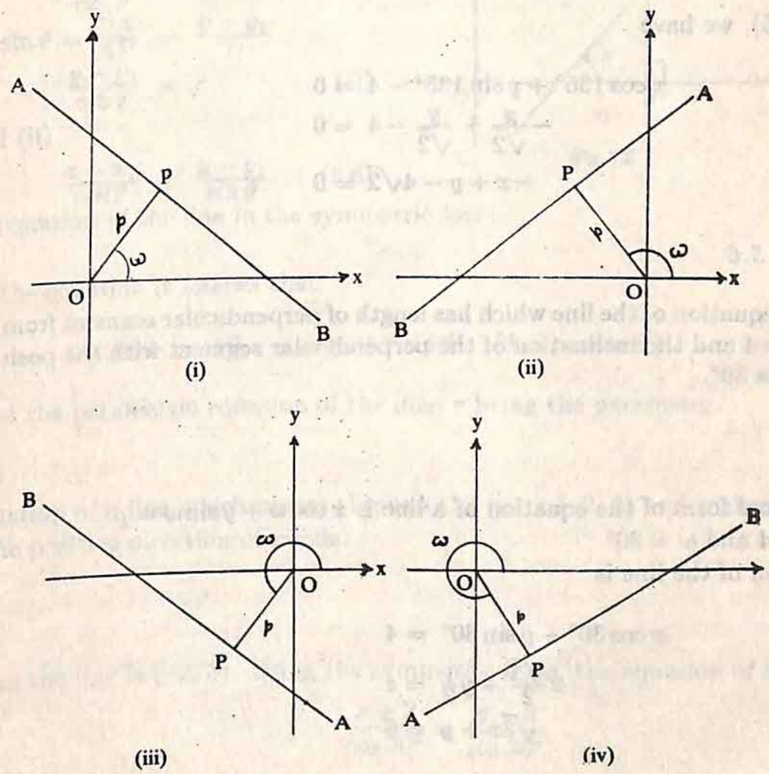


Fig.5.5

Let ω be the angle between OP and the positive x-axis, and p the length of the perpendicular OP. Then the coordinates of the point p will be $(p\cos\omega, p\sin\omega)$, and the slope of AB will be $-\frac{1}{\tan\omega} = -\cot\omega$. If (x, y) is any other point on the line AB, then by the point-slope formula, $y - p\sin\omega = -\cot\omega(x - p\cos\omega)$. On simplifying, we get

$$x\cos\omega + y\sin\omega - p = 0$$

or

$$x\cos\omega + y\sin\omega = p \tag{5.5}$$

which is the perpendicular form of the straight line. It can be verified that we get the same form, if we consider Fig. 5.5 (ii) (iii) and (iv).

Example 5.5

Find the equation of the line with $\omega = 135°$ and perpendicular distance 4.

Solution

From (5.5), we have

$$x\cos 135° + y\sin 135° - 4 = 0$$

or
$$-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - 4 = 0$$

or
$$-x + y - 4\sqrt{2} = 0$$

Example 5.6

Find the equation of the line which has length of perpendicular segment from the origin to the line as 4 and the inclination of the perpendicular segment with the positive direction of x-axis is 30°.

Solution

The normal form of the equation of a line is $x\cos\omega + y\sin\omega = p$
Here $p = 4$ and $\omega = 30°$
$\therefore$ Equation of the line is

$$x\cos 30° + y\sin 30° = 4$$

or
$$x\frac{\sqrt{3}}{2} + y\frac{1}{2} = 4$$

or
$$\sqrt{3}x + y = 8$$

5.7 Symmetric Form

Let a line pass through $A(x_1, y_1)$ and be inclined at an angle θ with the positive direction of x-axis. Then the equation of the line involving x_1, y_1 and θ is called the symmetric form of the line.

Let $P(x,y)$ be any point and $AP = r$. Draw AB and PC perpendiculars to x-axis from A and P respectively and $AN \perp PC$.

Now, $AN = BC = OC - OB = x - x_1$
$PN = PC - CN = PC - AB = y - y_1$
Also $AP = r$

In $\triangle ANP$, $\cos\theta = \frac{AN}{AP} = \frac{(x - x_1)}{r}$

i.e. $\frac{x - x_1}{\cos\theta} = r$ (i)

and $\sin\theta = \frac{PN}{AP} = \frac{y - y_1}{r}$

i.e. $\frac{y - y_1}{\sin\theta} = r$ (ii)

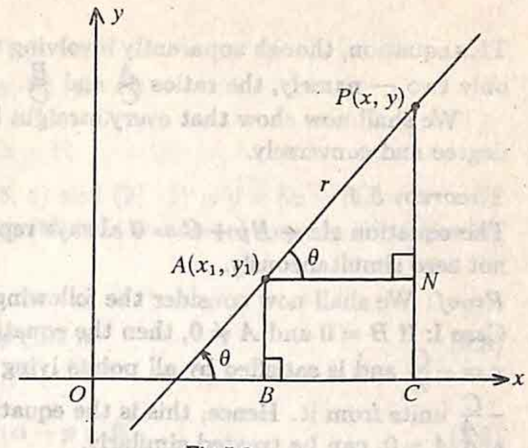


Fig.5.6

From (i) and (ii)

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} \quad (5.6)$$

which is the equation of the line in the symmetric form.

Note: From the equation it follows that

$$x = x_1 + r\cos\theta, \quad y = y_1 + r\sin\theta, \quad (\theta = \text{constant})$$

which is called the parametric equation of the line, r being the parameter.

Example 5.7

Find the equation of a line which passes through the point $(-2, 3)$ and makes an angle of $30°$ with the positive direction of x-axis.

Solution

Here $\theta = 30°$.
Given point on the line is $(-2, 3)$. Using the symmetric form, the equation of the line is

$$\frac{x+2}{\cos 30°} = \frac{y-3}{\sin 30°} \quad (5.7)$$

or $$\frac{x+2}{\frac{\sqrt{3}}{2}} = \frac{y-3}{\frac{1}{2}}$$

or $$\sqrt{3}y - 3\sqrt{3} = x + 2$$

or $$x - \sqrt{3}y + (3\sqrt{3} + 2) = 0$$

is the equation of the required line.

5.8 General Form

All the forms in which we have found the equation of the straight line are of the first degree in x and y. The converse of this is also true. The most general form of any equation of the first degree in x and y is $Ax + By + C = 0$.

This equation, though apparently involving three constants A, B and C, in reality involves only two — namely, the ratios $\frac{A}{C}$ and $\frac{B}{C}$.

We shall now show that every straight line is represented by an equation of the first degree and conversely.

Theorem 5.1

The equation $Ax + By + C = 0$ always represents a straight line provided A and B are not zero simultaneously.

Proof: We shall now consider the following two cases:

Case I: If $B = 0$ and $A \neq 0$, then the equation $Ax + By + C = 0$ becomes $Ax + C = 0$ or $x = -\frac{C}{A}$ and is satisfied by all points lying on a line parallel to y-axis and at a distance $-\frac{C}{A}$ units from it. Hence, this is the equation of a straight line. The case where $B \neq 0$, and $A = 0$, can be treated similarly.

Case II: If $B \neq 0$ and $A \neq 0$, we can solve the equation for y and obtain

$$y = -\frac{A}{B}x - \frac{C}{B}$$

This represents the straight line with slope $-\frac{A}{B}$ and y-intercept $-\frac{C}{B}$.

The converse is given in the following theorem.

Theorem 5.2

Every straight line has an equation of the form $Ax + By + C = 0$, where A, B and C are constants.

Proof: Given a straight line, either it cuts the y-axis, or is parallel to or coincident with it. We know that the equation of a line which cuts the y-axis (that is it has a y-intercept) can be put in the form $y = mx + b$; further, if the line is parallel or coincident with the y-axis, its equation is of the form $x = x_1$, where $x_1 = 0$ in the case of coincidence. Both of these equations are of the form given in the theorem; hence the proof.

We can use the form $Ax + By + C = 0$ to determine the equation of a straight line in the following way:

Example 5.8

Find the equation of the line through (3,4) and (2,–1).

Solution

We seek the numbers A, B and C such that the line $Ax + By + C = 0$ passes through the two given points. If $Ax + By + C = 0$ passes through (3,4), then $3A + 4B + C = 0$; if it passes through $(2, -1)$, then $2A - B + C = 0$.

Subtracting $2A - B + C = 0$ from $3A + 4B + C = 0$, we get $A + 5B = 0$.

So $A = -5B$; also $2A - B + C = 0$ now yields $-10B - B + C = 0$, i.e. $C = 11B$.

Thus the equation $Ax+By+C=0$ becomes

$$-5Bx+By+11B=0$$
$$\text{or}\quad -5x+y+11=0$$
$$\text{or}\quad y=5x-11$$

Therefore, the equation of the line through (3, 4) and (2, –1) is $y=5x-11$.

The general equation of a straight line is reducible to the normal form in the following way

The general equation of the straight line is

$$Ax+By+C=0 \tag{5.8}$$

The equation of a line in the normal form is

$$x\cos\alpha+y\sin\alpha-p=0 \tag{5.9}$$

If we suppose that (5.8) and (5.9) represent the same straight line, then we can compare the corresponding coefficients

$$\frac{A}{\cos\alpha}=\frac{B}{\sin\alpha}=\frac{C}{-p}$$

or $$\cos\alpha=-\frac{Ap}{C}\quad\text{and}\quad\sin\alpha=-\frac{Bp}{C}$$

Squaring and adding both, we have

$$\cos^2\alpha+\sin^2\alpha=\frac{A^2p^2}{C^2}+\frac{B^2p^2}{C^2}$$

or $$1=\frac{p^2}{C^2}(A^2+B^2)$$

or $$p^2=\frac{C^2}{A^2+B^2}$$

or $$p=\pm\frac{C}{\sqrt{A^2+B^2}}$$

But p is the measure of the perpendicular segment, and is, therefore, taken to be positive.

Assume that $C\geq 0$ without loss of generality.

Hence, $p=\dfrac{C}{\sqrt{A^2+B^2}}$

Therefore, $\cos\alpha=-\dfrac{A}{\sqrt{A^2+B^2}}$

and $\sin\alpha=-\dfrac{B}{\sqrt{A^2+B^2}}$

Hence (5.9) takes the form

$$-\frac{A}{\sqrt{A^2+B^2}}x-\frac{B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}}$$

which is the required equation in normal form.

5.9 Angle between Two Lines

We shall consider any two non-perpendicular lines ℓ_1 and ℓ_2, neither of which is parallel to the y-axis and derive a formula for the angle from ℓ_1 to ℓ_2 in terms of their slopes.

The angle between the lines ℓ_1 and ℓ_2 is either acute or obtuse. (See Fig. 5.7) From Fig. 5.7 (i) we see that

$$\alpha_2 = \alpha_1 + \theta,$$

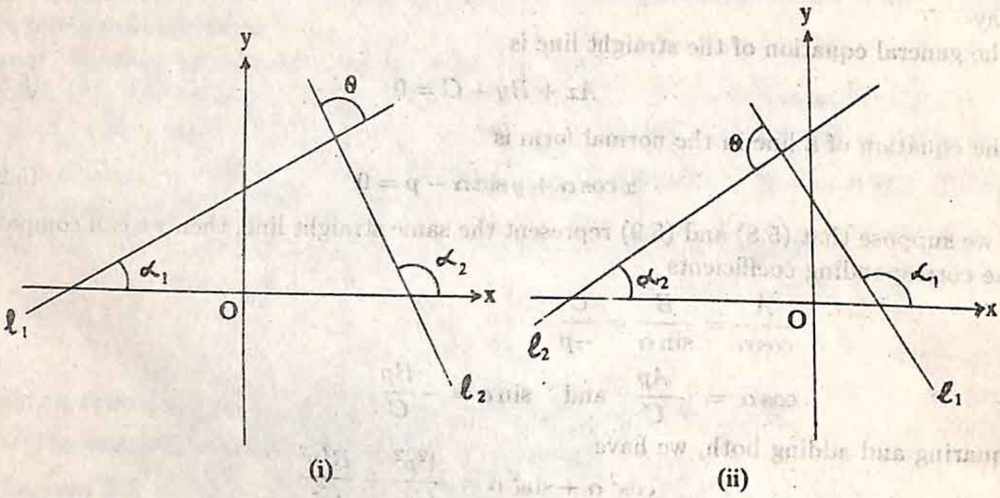


Fig. 5.7

since α_2 is an exterior angle of a triangle with α_1 and θ as the opposite interior angles. Therefore,

$$\theta = \alpha_2 - \alpha_1$$

and

$$\tan\theta = \tan(\alpha_2 - \alpha_1)$$

$$= \frac{\tan\alpha_2 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2}$$

Therefore,

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \tag{5.10}$$

where $m_1 = \tan\alpha_1$, and $m_2 = \tan\alpha_2$

From Fig. 5.7(ii), we see that

$$\alpha_1 = \alpha_2 + (\pi - \theta),$$

since $(\pi - \theta)$ and α_2 are interior angles with α_1 as the opposite exterior angle.

Therefore, $\theta = \alpha_2 - \alpha_1 + \pi$

or $\tan\theta = \tan(\pi + (\alpha_2 - \alpha_1))$

$$= \tan(\alpha_2 - \alpha_1)$$

$$= \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_2 \tan\alpha_1}$$

Thus $\quad \tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

We have thus proved the following theorem.

Theorem 5.3

The positive angle θ from the line ℓ_1 to the line ℓ_2 with slopes $m_1 = \tan\alpha_1$ and $m_2 = \tan\alpha_2$ respectively is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

We note that if two lines are perpendicular to each other, then, as seen earlier,

$$m_1 = -\frac{1}{m_2}, \quad \text{that is} \quad 1 + m_1 m_2 = 0$$

Thus, $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$ is not defined in this case.

Notice that in numerical examples, the value of $\tan\theta$ will sometimes be found to be negative. This would merely mean that instead of getting the acute angle of intersection, its supplement, which too is the angle of intersection of the lines, is being obtained.

Example 5.9

Determine the angle B of the triangle with vertices $A(-2,1), B(2,3)$ and $C(-2,-4)$.

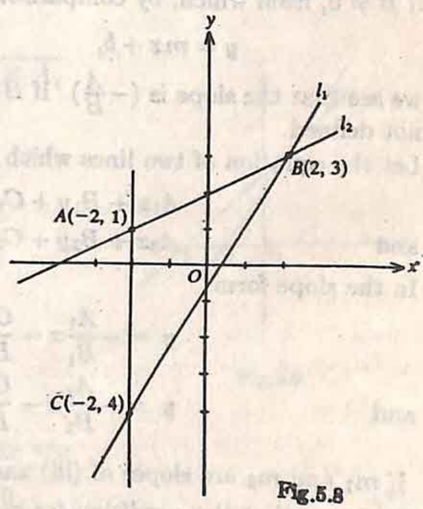


Fig.5.8

Solution

Let BA be ℓ_1, BC be ℓ_2 so that formula 5.10 will give the desired angle (See Fig. 5.8).

Then,
$$m_1 = \frac{1-3}{-2-2} = \frac{1}{2}$$

$$m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

and
$$\tan\angle B = \frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4}\,\frac{1}{2}} = \frac{\frac{5}{4}}{\frac{15}{8}} = \frac{2}{3} = 0.667$$

Thus $\angle B \simeq 33°42'$.

Example 5.10

Find the value of m_1 if $m_2 = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$.

Solution

From the formula

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2},$$

we get

$$\tan\frac{\pi}{4} = \frac{\frac{1}{2} - m_1}{1 + \frac{m_1}{2}}$$

or

$$1 = \frac{1 - 2m_1}{2 + m_1}$$

or

$$2 + m_1 = 1 - 2m_1$$

Therefore,

$$m_1 = -\frac{1}{3}$$

Condition for Perpendicularity

The equation $Ax + By + C = 0$ may be written in the form

$$y = -\frac{A}{B}x - \frac{C}{B}$$

if $B \neq 0$, from which, by comparison with the form

$$y = mx + b,$$

we see that the slope is $(-\frac{A}{B})$. If $B = 0$, the line is parallel to the y-axis and its slope is not defined.

Let the equation of two lines which are not prallel to the y-axis be

$$A_1x + B_1y + C_1 = 0 \qquad \text{(i)}$$

and

$$A_2x + B_2y + C_2 = 0 \qquad \text{(ii)}$$

In the slope form

$$y = -\frac{A_1}{B_1}x - \frac{C_1}{B_1} \qquad \text{(iii)}$$

and

$$y = -\frac{A_2}{B_2}x - \frac{C_2}{B_2} \qquad \text{(iv)}$$

If m_1 and m_2 are slopes of (iii) and (iv), then $m_1 = \frac{-A_1}{B_1}$ and $m_2 = \frac{-A_2}{B_2}$

We know that the condition for perpendicularity of two lines whose slopes are m_1 and m_2 is $m_1 m_2 = -1$

This means

$$\left(\frac{-A_1}{B_1}\right)\left(\frac{-A_2}{B_2.}\right) = -1$$

or
$$\frac{A_1A_2}{B_1B_2} = -1$$
or
$$A_1A_2 + B_1B_2 = 0 \tag{5.11}$$
which is the condition for perpendicularity of the lines given in the general form.

Note: If $B_1 = 0$, any line parallel to the x-axis will be perpendicular to the line $Ax + By + C = 0$. In this case too (5.11) is satisfied since $A_2 = 0$.

5.10 Condition for Concurrency of Three Straight Lines

Let
$$A_1x + B_1y + C_1 = 0 \tag{i}$$
and
$$A_2x + B_2y + C_2 = 0. \tag{ii}$$
be the two straight lines AL_1 and AL_2 respectively intersecting each other at the point A as shown in Fig. 5.9.

Since (i) is the equation of AL_1, the coordinates of any point on it must satisfy the equation (i). Similarly, the coordinates of any point on AL_2 must satisfy (ii).

Now, the only point which is common to both the straight lines is their point of intersection A. The coordinates of this point must, therefore, satisfy both (i) and (ii). If (x_1, y_1) are the coordinates of A, then we have
$$A_1x_1 + B_1y_1 + C_1 = 0 \tag{iii}$$
$$A_2x_1 + B_2y_1 + C_2 = 0. \tag{iv}$$

Solving (iii) and (iv), we have
$$\frac{x_1}{B_1C_2 - B_2C_1} = \frac{y_1}{C_1A_2 - A_1C_2} = \frac{1}{A_1B_2 - B_1A_2}$$

This means
$$x_1 = \frac{B_1C_2 - B_2C_1}{A_1B_2 - B_1A_2}$$
and
$$y_1 = \frac{C_1A_2 - A_1C_2}{A_1B_2 - B_1A_2}$$

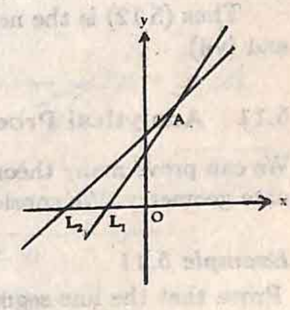


Fig.5.9

Hence, the point of intersection of the two lines (i) and (ii) is
$$\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - B_1A_2}, \frac{C_1A_2 - A_1C_2}{A_1B_2 - B_1A_2}\right)$$
Now the condition that the three lines whose equations are
$$A_1x + B_1y + C_1 = 0 \tag{v}$$
$$A_2x + B_2y + C_2 = 0 \tag{vi}$$
$$A_3x + B_3y + C_3 = 0 \tag{vii}$$

should be concurrent is that the point of intersection of (v) and (vi) must lie on (vii). In other words, the coordinates of the point of intersection of (v) and (vi) should satisfy the equation of (vii) i.e.

$$A_3\left(\frac{B_1C_2-B_2C_1}{A_1B_2-A_2B_1}\right)+B_3\left(\frac{C_1A_2-C_2A_1}{A_1B_2-A_2B_1}\right)+C_3=0$$

or
$$A_3(B_1C_2-B_2C_1)+B_3(C_1A_2-C_2A_1)+C_3(A_1B_2-A_2B_1)=0$$

which is the required condition for concurrency.

This condition can be expressed in the determinant form as

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}=0 \tag{5.12}$$

We have proved that if the lines (v), (vi), (vii) are concurrent then (5.12) is true.

Conversely, if (5.12) is true, then it would imply that

$$A_3\left(\frac{B_1C_2-B_2C_1}{A_1B_2-A_2B_1}\right)+B_3\left(\frac{C_1A_2-C_2A_1}{A_1B_2-A_2B_1}\right)+C_3=0$$

i.e. the point

$$\left(\frac{B_1C_2-B_2C_1}{A_1B_2-A_2B_1},\frac{C_1A_2-C_2A_1}{A_1B_2-A_2B_1}\right),$$

which, as we have seen, is the point of intersection of (v) and (vi), lies on (vii). Thus if (5.12) is true then (v), (vi), (vii) are concurrent.

Thus (5.12) is the necessary and sufficient condition for the concurrency of (v), (vi) and (vii).

5.11 Analytical Proofs of Geometric Theorems

We can prove many theorems of plane geometry with the help of the formulae of coordinate geometry. We consider some examples.

Example 5.11

Prove that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to one-half its length.

Solution

Consider a triangle OPQ with vertices at $O(0,0)$, $P(a,0)$ and $Q(b,c)$. (See Fig. 5.10) The mid-point A of OQ is $\left(\frac{b}{2},\frac{c}{2}\right)$. The mid-point B of PQ is $\left(\frac{a+b}{2},\frac{c}{2}\right)$. The slope m_1 of AB is given by

$$m_1 = \frac{\frac{c}{2} - \frac{c}{2}}{\left(\frac{a+b}{2} - \frac{b}{2}\right)} = 0$$

The slope m_2 of OP is given by

$$m_2 = \frac{0-0}{a-0} = 0$$

Thus $m_1 = m_2$.

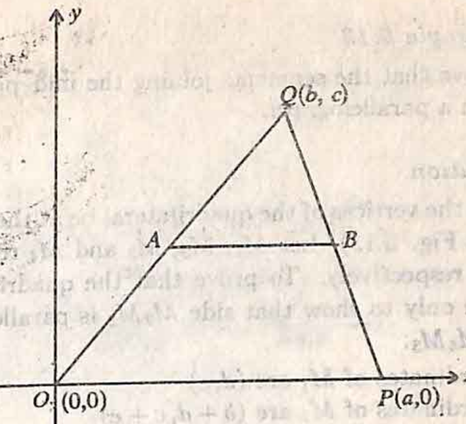


Fig.5.10

Hence, the line joining the mid-points of the two sides is parallel to the third side.

Also, $$AB = \sqrt{\left(\frac{a+b}{2} - \frac{b}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2} = \frac{a}{2},$$

and $$OP = \sqrt{(0-a)^2 + (0-0)^2} = a$$

Hence, the line joining the two mid-points A and B is equal to one-half of the base.

Example 5.12

Prove that the diagonals of a parallelogram bisect each other.

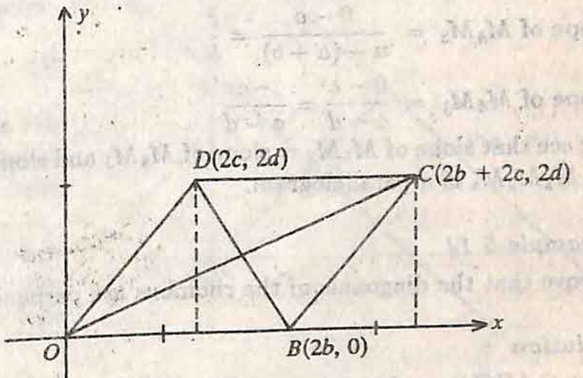


Fig.5.11

Solution

To prove that the diagonals bisect each other, it is only necessary to show that the mid-point of each diagonal is the same point. Let $OBCD$ be the parallelogram with three of its vertices at the points $O(0,0)$, $B(2b,0)$ and $D(2c,2d)$ (See Fig. 5.11). It may be seen that the fourth vertex is $C(2b+c,2d)$. The mid-point of both the diagonals OC and BD is $(b+c,d)$.

This completes the proof.

Example 5.13

Prove that the segments joining the mid-points of the adjacent sides of a quadrilateral form a parallelogram.

Solution

Let the vertices of the quadrilateral be at the points $O(0,0), A(2a,0), B(2b,2c), C(2d,2e)$ (See Fig. 5.12). Let M_1, M_2, M_3 and M_4 represent the mid-points of OC, CB, BA and OA respectively. To prove that the quadrilateral $M_1M_2M_3M_4$ is a parallelogram, we have only to show that side M_2M_3 is parallel to side M_1M_4, and side M_1M_2 is parallel to M_4M_3.

Coordinates of M_1 are (d, e)
Coordinates of M_2 are $(b+d, c+e)$
Coordinates of M_3 are $(a+b, c)$
Coordinates of M_4 are $(a, 0)$
We now find slopes of
M_1M_2, M_2M_3, M_4M_3 and M_4M_1

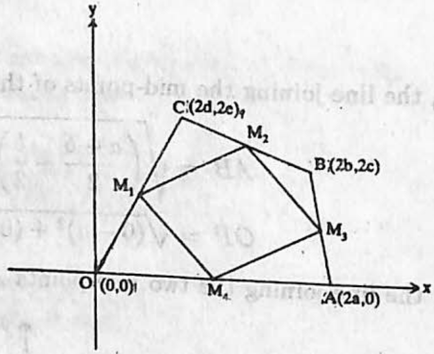


Fig.5.12

$$\text{Slope of } M_1M_2 = \frac{c+e-e}{b+d-d} = \frac{c}{b}$$

$$\text{Slope of } M_2M_3 = \frac{c-(c+e)}{a+b-(b+d)} = \frac{-e}{a-d}$$

$$\text{Slope of } M_4M_3 = \frac{0-c}{a-(a+b)} = \frac{c}{b}$$

$$\text{Slope of } M_4M_1 = \frac{0-e}{a-d} = \frac{-e}{a-d}$$

We see that slope of $M_1M_2 =$ slope of M_4M_3 and slope of $M_2M_3 =$ slope of M_4M_1. Thus, $M_1M_2M_3M_4$ is a parallelogram.

Example 5.14

Prove that the diagonals of the rhombus are perpendicular to each other.

Solution

Let $OABC$ be a rhombus with its vertices as $(0,0), (x_1, 0), (x_1+x_2, y_2)$ and (x_2, y_2). As $OABC$ is a rhombus, all of its sides are equal.

Hence, $\quad OA = OC$

or $\quad OA^2 = OC^2$

i.e. $\quad {x_1}^2 = {x_2}^2 + {y_2}^2$

To show that its diagonals are perpendicular to each other, we shall show that the product of the slopes of the diagonals is -1.

Now slope of $OB = \dfrac{y_2}{x_1 + x_2}$

and slope of $AC = \dfrac{y_2}{x_2 - x_1}$

Hence, the product of the slopes

$$= \frac{y_2}{x_1 + x_2} \cdot \frac{y_2}{x_2 - x_1}$$

$$= \frac{{y_2}^2}{{x_2}^2 - {x_1}^2}$$

$$= \frac{{y_2}^2}{-{y_2}^2} = -1 \quad \{\text{using (i)}\}$$

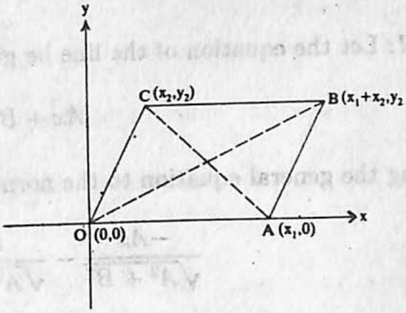


Fig.5.13

Hence, the diagonals of a rhombus are perpendicular to each other.

5.12 Distance of a Point from a Line

Case I: Let the equation of a line AB be

$$x \cos\alpha + y \sin\alpha = p \qquad \text{(i)}$$

Let P be the point (x', y') and d be the length of the perpendicular NP, P being assumed to lie on the opposite side of the line AB from O.

Draw $A'PB'$ parallel to AB through P, and draw the common perpendicular OTT' from O on AB and $A'B'$ (See Fig. 5.14).

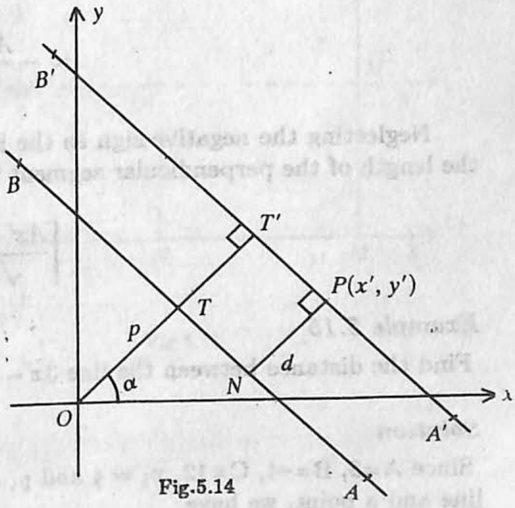


Fig.5.14

Then $OT = p$ and the angle $AOT = \alpha$.

The perpendicular from O on $A'B'$ is of length OT', and makes an angle α with OX; hence the equation of $A'B'$ is

$$x \cos\alpha + y \sin\alpha = p + d$$

The coordinates (x', y') of P satisfy the equation of $A'B'$

$$\therefore \quad x' \cos\alpha + y' \sin\alpha = p + d$$

or

$$d = x' \cos\alpha + y' \sin\alpha - p$$

Hence, the length of the perpendicular is the result of substituting the coordinates of P in the expression $x\cos\alpha + y\sin\alpha - p$.

Case II: Let the equation of the line be given in the general form

$$Ax + By + C = 0 \tag{ii}$$

Reducing the general equation to the normal form, we have

$$\frac{-Ax}{\sqrt{A^2+B^2}} - \frac{By}{\sqrt{A^2+B^2}} = \frac{C}{\sqrt{A^2+B^2}}$$

Now, the length of the perpendicular segment drawn from the given point (x', y') to (ii) is

$$\frac{-Ax' - By' - C}{\sqrt{A^2+B^2}}$$
$$= -\frac{Ax' + By' + C}{\sqrt{A^2+B^2}}$$

Neglecting the negative sign as the length of a segment is always positive, we have the length of the perpendicular segment as

$$\left|\frac{Ax' + By' + C}{\sqrt{A^2+B^2}}\right| \tag{5.13}$$

Example 5.15

Find the distance between the line $3x - 4y + 12 = 0$ and the point (4, 1).

Solution

Since A=3, B=-4, C=12, $x_1 = 4$ and $y_1 = 1$, by the formula for the distance between a line and a point, we have

$$d = \frac{|3\times 4 + (-4)\times 1 + 12|}{\sqrt{(3)^2 + (-4)^2}}$$
$$= \frac{|12 - 4 + 12|}{\sqrt{25}} = \frac{20}{5} = 4$$

Example 5.16

Find the perpendicular distance of the point (a, b) from the line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution

The required distance is

$$d = \frac{\frac{a}{a} + \frac{b}{b} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{ab}{\sqrt{a^2 + b^2}}$$

5.13 Translation of Axes

Let (x, y) be the coordinates of any point P referred to the axes OX and OY. Let $O'X'$ and $O'Y'$ be the new axes parallel to OX and OY. Let O' being the new origin. Let (h, k) be the coordinates of O' referred to the old axes OM and MP are the abscissa and ordinate of the point P referred to the old axes i.e. $OM = x$ and $MP = y$. $O'M' = x'$ and $M'P$ are the abscissa and ordinate of P referred to the new axes $O'X'$ and $O'Y'$ Let $O'M' = x'$ and $M'P = y'$. If we want to transform an equation in x and y into corresponding equation in x' and y', we will have to express the old coordinates x, y in terms of the new ones x', y' so that the transformation can be performed by direct substitution. It is easily seen from the figure that

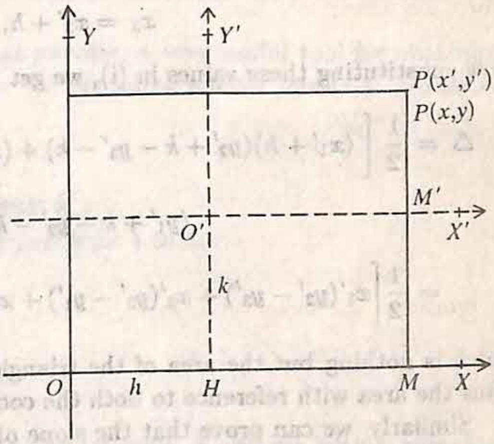


Fig.5.15

$$OM = h + O'M'; \; MP = k + M'P,$$

or

$$x = x' + h, \quad y = y' + k \tag{5.14}$$

We have, therefore, to write $(x' + h)$ for x and $(y' + k)$ for y in the equation which we wish to transform. We thus get an equation in x', y'. So if the equation of the set of points P (locus of P) with respect to OX and OY be $f(x, y) = 0$, the equation to the same set of points when the origin is transferred to O', the axes retaining their directions, becomes $f(x' + h, y' + k) = 0$ where x', y' are the current coordinates with reference to the new axes.

We can easily check that the area of a triangle (or a quadrilateral) and a slope of a line remain unaffected with the change of axes, i.e. the area of a triangle and slope of a line are invariant under the translation of axes.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC referred to some rectangular coordinate axes. Let the origin be shifted to the point (h, k) with axes retaining their directions. Then the area Δ of the triangle ABC with respect to the old coordinate axes is given by

$$\Delta = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right] \tag{i}$$

Let (x_1', y_1'), (x_2', y_2') and (x_3', y_3') be the coordinates of A, B, C, with respect to the new axes. Then we know that

$$\begin{aligned} x_1 &= x_1' + h, \; y_1 = y_1' + k \\ x_2 &= x_2' + h, \; y_2 = y_2' + k \\ x_3 &= x_3' + h, \; y_3 = y_3' + k \end{aligned}$$

Now substituting these values in (i), we get

$$\begin{aligned} \Delta &= \frac{1}{2}\Big[(x_1' + h)(y_2' + k - y_3' - k) + (x_2' + h)(y_3' + k - y_1' - k) + (x_3' + h) \times \\ &\qquad (y_1' + k - y_2' - k)\Big] \\ &= \frac{1}{2}\left[x_1'(y_2' - y_3') + x_2'(y_3' - y_1') + x_3'(y_1' - y_2')\right] \end{aligned}$$

which is nothing but the area of the triangle with respect to the new coordinate axes. Thus the area with reference to both the coordinate axes remains the same.

Similarly, we can prove that the slope of a line does not change with the change of axes (parallel translation).

For, if

$$Ax + By + C = 0 \tag{ii}$$

is the equation of a straight line referred to the old coordinate axes, then the slope of the line is seen to be equal to

$$m = -\frac{A}{B}$$

Now, if the origin is shifted to the point (h, k), then as before any point (x', y') on the straight line with respect to the new coordinate axes will satisfy the following relations.

$$x = x' + h \text{ and } y = y' + k$$

Therefore, the equation of the straight line can be written as

$$A(x' + h) + B(y' + k) + C = 0$$

or

$$Ax' + By' + (Ah + Bk + C) = 0 \tag{iii}$$

which is the equation of the straight line in the new coordinate axes. Now, we can easily see that the slope of the straight line given by (iii) is

$$m' = -\frac{A}{B}$$

which is the same as m. Therefore, we have established that the slope of a straight line is invariant under the translation of axes.

In fact, as we have learned earlier, all the geometric properties in Euclidean Geometry remain unchanged under rigid motion. Since parallel translation of axes is only a particular case of rigid motion (which includes rotation also), it is quite obvious that all geometric properties shall remain unchanged when we transform the coordinates in this way.

We shall see later that translation of axes provides a very useful tool for obtaining equations of different loci in simple form, or for providing simple proofs of geometric properties.

EXERCISE 5.1

Find the equation of the line in each of the problems 1 to 4.

1. Through (4, 3) and slope 2.
2. Through (0,–2) with slope –4.
3. Through (0, –3) and (5, 0).
4. Through (–1, –2) and (–5, –2).
5. Find the lines through the point (0, 2) making an angle $\frac{\pi}{2}$ and $\frac{2\pi}{3}$ with x-axis. Also find the lines parallel to them cutting the y-axis at a distance 2 below the origin. Find their point of intersection with x-axis.
6. What are the inclinations to the x-axis of the lines

$$y = \frac{1}{3}x\sqrt{3} + 3 \text{ and } y = \sqrt{3}x + 3?$$

   Show that the line $y = x + 3$ bisects the angle between them.
7. Find the equation of the line that has y-intercept 4 and is parallel to the line $2x - 3y = 7$.
8. Find the equation of the line that has x-intercept –3 and is perpendicular to the line $3x + 5y = 4$.
9. Find the equation of a straight line passing through the point (2, 2), such that the sum of its intercepts on the axes is 9.
10. The vertices of a triangle are the points (0, 0), (2, 4) and (6, 4). Find the equations of its sides.

11. The mid-points of the sides of a triangle are (2, 1), (–5, 7), (–5, –5). Find the equations of the sides.
12. Find the equation of the line that is parallel to $2x + 5y = 7$ and passes through the mid-point of the line joining (2, 7) and (–4, 1).
13. Find the equation of the line that is perpendicular to $3x + 2y = 8$ and passes through the mid-point of the line joining (5, –2) and (2, 2).
14. Find the equation of the straight line bisecting the segment joining the points (5, 3) and (4, 4) and making an angle of 45° with the x-axis.
15. If p be the measure of the perpendicular segment from the origin on the line whose intercepts on the axes are a and b, show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

16. Obtain the perpendicular form of the equation of lines from the given values of p and ω,

    (i) $p = 3,\ \omega = 45°$ (ii) $p = 5,\ \omega = 30°$
    (iii) $p = 5,\ \omega = 135°$ (iv) $p = 1,\ \omega = 90°$

17. Reduce each of the following to the perpendicular form and find p.

    (i) $x + y - 2 = 0$ (ii) $4x + 3y - 9 = 0$
    (iii) $x - 4 = 0$ (iv) $y - 2 = 0$

18. Which of the lines $2x - y + 3 = 0$ and $x - 4y - 7 = 0$ is farther from the origin?
19. The perpendicular distance of a line from the origin is 5 cm and its slope is –1. Find the equation of the line.
20. Show that the origin is equidistant from the three straight lines, $4x + 3y + 10 = 0,\ 5x - 12y + 26 = 0,\ 7x + 24y = 50$
21. Find the angle between the straight lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.
22. Find the equation of the lines through the origin making angle of 60° with the line $x + y\sqrt{3} + 3\sqrt{3} = 0$, also the coordinates of points where they meet the line.

Classify the pairs of lines in Exercise 23 to 25 as coincident, parallel, or intersecting.

23. $6x + 14y - 16 = 0,\quad 12x + 28y - 32 = 0$
24. $3x - 4y = 8,\quad 3x + 4y = 11$
25. $x - 2y = 7,\quad 4y - 2x = 13$

Find the distance between the line and the point in each of the following Exercises from 26 to 28.

26. $4x + 3y - 5 = 0.\quad (-2, -1)$
27. $5x + 12y - 41 = 0,\quad (3, 0)$
28. $y = 4,\quad (2, 3)$

CHAPTER 6

Family of Lines

6.1 Equation of Family of Lines

Let $\quad A_1x + B_1y + C_1 = 0 \quad (6.1)$

and $\quad A_2x + B_2y + C_2 = 0 \quad (6.2)$

be two given non-parallel lines. Then, for any real number k,

$$(A_1x + B_1y + C_1) + k(A_2x + B_2y + C_2) = 0 \tag{6.3}$$

for any values of x and y for which (6.1), (6.2) are both true. In other words the point of intersection (x_0, y_0) of (6.1) and (6.2) lies on the straight line whose equation is (6.3). Hence (6.3) represents a line through the point of intersection of (6.1) and (6.2) for every k. Hence, (6.3) represents the *family of lines* passing through the point of intersection of the two lines.

Note: If there exists no point (x_0, y_0) common to the lines (6.1) and (6.2), they are parallel. In this case (6.3) gives the family of lines parallel to them.

Example 6.1

Find the equation of the straight line parallel to the y-axis and drawn through the point of intersection of $x - 7y + 5 = 0$ and $3x + y - 7 = 0$.

Solution

The equation of any straight line through the point of intersection of the given lines is of the form

$$x - 7y + 5 + k(3x + y - 7) = 0$$

or

$$(1 + 3k)x + (k - 7)y + 5 - 7k = 0$$

If this line is parallel to y-axis, the coefficient of y is zero, hence $k = 7$, and the equation becomes

$$x - 2 = 0$$

Example 6.2

Find the equation of the line through the intersection of $3x+4y=7$ and $x-y+2=0$, and with slope 5.

Solution

The equation of the family of lines passing through the point of intersection of the given lines will be

$$(3x+4y-7)+k(x-y+2)=0$$

or

$$(3+k)x+(4-k)y+(2k-7)=0$$

The slope of each member of the family is

$$\frac{k+3}{k-4}$$

But slope for the line is given to be 5. Hence,

$$\frac{k+3}{k-4}=5$$

or

$$k+3=5k-20, \text{ i.e., } \quad k=\frac{23}{4}$$

Therefore, the equation of the desired line is

$$(3x+4y-7)+\frac{23}{4}(x-y+2)=0$$

or

$$4(3x+4y-7)+23(x-y+2)=0$$

or

$$35x-7y+18=0$$

EXERCISE 6.1

1. Find the equation of the line through the point of intersection of $x+2y=5$ and $x-3y=7$, and passing through the point
   (i) (0,0) (ii) (2,–3)
   (iii) (1,0) (iv) (0,–1)
2. Find the equation of the line through the point of intersection of $5x-3y=1$ and $2x+3y=23$, and perpendicular to the line whose equation is
   (i) $x-2y=3$ (ii) $x=0$
   (iii) $y=0$ (iv) $5x-3y=1$
3. Find the equation of the line through the intersection of the lines $x+2y-3=0$ and $4x-y+7=0$ and which is parallel to $5x+4y-20=0$
4. Find the equation of the line through the intersection of the lines $2x+3y-4=0$ and $x-5x+7=0$ that has its x-intercept equal to –4

6.2 Pair of Straight Lines through Origin

We shall show that the homogeneous equation of the second degree

$$ax^2 + 2hxy + by^2 = 0$$

represents a pair of straight lines passing through the origin if $h^2 \geq ab$.
Solving the above equation as a quadratic equation for x, we get

$$x = \frac{-h \pm \sqrt{h^2 - ab}}{a} \quad y, (h^2 \geq ab)$$

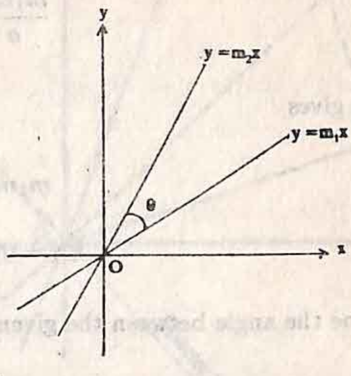


Fig.6.1

i.e., either $\quad ax + (h + \sqrt{h^2 - ab})y = 0$

or $\quad ax + (h - \sqrt{h^2 - ab})y = 0.$

Each of the above is a straight line passing through the origin.
Hence, the homogenous equation

$$ax^2 + 2hxy + by^2 = 0$$

represents two straight lines passing through the origin. These lines are real and distinct, if $h^2 > ab$ and coincident if $h^2 = ab$, and the lines do not exist if $h^2 < ab$.

6.3 Angle between the Pair of Straight Lines

Let the pair of straight lines be represented by the equation $ax^2 + 2hxy + by^2 = 0$.

Since the above equation represents a pair of straight lines passing through the origin, they will be given by the equations

$$y = m_1x \quad (6.4)$$

$$y = m_2x \quad (6.5)$$

and $\quad y - m_1x = 0 \quad \text{and} \quad y - m_2x = 0$

or $\quad (y - m_1x)(y - m_2x) = 0$

Then

or $\quad m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$

This equation is the same as

$$ax^2 + 2hxy + by^2 = 0$$

---

[1] The equation $ax^2 + 2hxy + by^2 = 0$ is called a homogeneous equation of second degree in x and y as the degree of each of the terms $ax^2, 2hxy, by^2$ is 2 and it does not contain any other terms.

Therefore comparing the coefficients,

$$\frac{m_1 m_2}{a} = -\frac{(m_1 + m_2)}{2h} = \frac{1}{b}$$

This gives

$$m_1 m_2 = \frac{a}{b} \tag{6.6}$$

and

$$m_1 + m_2 = -\frac{2h}{b} \tag{6.7}$$

If θ be the angle between the given straight lines, then

$$\begin{aligned}\tan\theta &= \frac{m_2 - m_1}{1 + m_1 m_2} \\ &= \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}\end{aligned}$$

Substituting the values of $m_1 m_2$ and $m_1 + m_2$, we obtain

$$\tan\theta = \frac{\sqrt{4h^2 - 4ab}}{a+b} \tag{6.8}$$

$$\text{Hence,}\quad \theta = \tan^{-1}\left\{\frac{2\sqrt{h^2 - ab}}{a+b}\right\} \tag{6.9}$$

Notice that the condition for coincidence of the lines is

$$h^2 = ab \tag{6.10}$$

In other words, for the coincidence of lines, the expression $ax^2 + 2hxy + by^2$ should be a perfect square.

The condition for perpendicularity is

$$a + b = 0 \tag{6.11}$$

6.4 Equations of the Bisectors of the Angles

Let the lines be represented by the equation

$$ax^2 + 2hxy + by^2 = 0$$

Suppose the lines are

$$y - m_1x = 0, \;\; y - m_2x = 0$$

Since from any point on a bisector the perpendicular distances on the lines are equal, we have

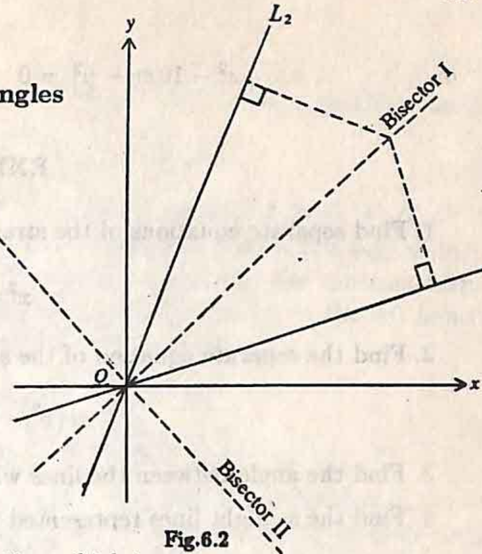


Fig.6.2

$$\frac{y - m_1x}{\sqrt{1+m_1^2}} = \pm\frac{y - m_2x}{\sqrt{1+m_2^2}}$$

The two bisectors can be expressed in one equation which is

$$\left(\frac{y - m_1x}{\sqrt{1+m_1^2}} + \frac{y - m_2x}{\sqrt{1+m_2^2}}\right)\left(\frac{y - m_1x}{\sqrt{1+m_1^2}} - \frac{y - m_2x}{\sqrt{1+m_2^2}}\right) = 0$$

or
$$(1+m_2^2)(y-m_1x)^2 - (1+m_1^2)(y-m_2x)^2 = 0$$

or
$$x^2(m_1^2 - m_2^2) - 2xy(m_1 - m_2)(1 - m_1m_2) + y^2(m_2^2 - m_1^2) = 0$$

or
$$x^2 - y^2 = 2xy\frac{1 - m_1m_2}{m_1 + m_2} \quad (\text{since } m_1 - m_2 \neq 0)$$

Substituting $\frac{a}{b}$ for m_1m_2 and $-\frac{2h}{b}$ for $m_1 + m_2$ [see (6.6) and (6.7)], the above equation becomes

$$x^2 - y^2 = 2xy\frac{a-b}{2h} \tag{6.12}$$

i.e.
$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \tag{6.13}$$

which is the required equation of the bisectors of the angles between the pair of lines given by $ax^2 + 2hxy + by^2 = 0$.

Example 6.3

Find the equation of the lines bisecting the angles between the pair of lines $3x^2 + xy - 2y^2 = 0$.

Solution

Using the formula $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$, we get the equation

$$\frac{x^2 - y^2}{3 - (-2)} = \frac{xy}{\frac{1}{2}}$$

or $$x^2 - 10xy - y^2 = 0$$

EXERCISE 6.2

1. Find separate equations of the straight lines whose joint equation is

$$x^2 - 5xy + 6y^2 = 0$$

2. Find the separate equation of the straight lines whose joint equation is

$$ab(x^2 - y^2) + (a^2 - b^2)xy = 0$$

3. Find the angle between the lines whose joint equation is $2x^2 - 3xy - 6y^2 = 0$.
4. Find the straight lines represented by the equation

$$y^2 - xy - 6x^2 = 0$$

and find the angle between them.
5. Prove that the angle between the straight lines given by

$$(x\cos\alpha - y\sin\alpha)^2 = (x^2 + y^2)\sin^2\alpha$$

is 2α.
6. Show that the bisectors of the angles between the lines

$$(ax + by)^2 = c(bx - ay)^2$$

are respectively parallel and perpendicular to the line $ax + by + c = 0$.

6.5 Condition for the General Equation of Second Degree to Represent Two Straight Lines

We have seen that the general equation of the first degree in x and y, viz., $Ax+By+C=0$ represents a straight line. Let us now consider the product equation.

$$(Ax + By + C)(A'x + B'y + C') = 0 \tag{6.14}$$

and examine what locus is represented by this.

Since the two factors on the left hand side of equation (6.14) are $Ax + By + C$ and $A'x + B'y + C'$, the equation is obviously satisfied if either of these factors is equal to zero. Hence, equation (6.14) represents a pair of straight lines

$$Ax + By + C = 0 \quad \text{and} \quad A'x + B'y + C' = 0.$$

If we multiply the two factors on the left hand side of (6.14), we get an equation of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{6.15}$$

where the values of a, b, c, f, g, h are easily determined in terms of A, B, C, A', B', C'. This is the most general equation of the second degree and will represent a pair of straight lines provided the coefficients of various powers of x and y and the constant term are suitably determined. The obvious condition is that the expression on the left hand side of equation (6.15) should break up into linear factors in x and y.

Let the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent a pair of straight lines and let (x_1, y_1) be the point of intersection of the lines.

If the origin is transferred to (x_1, y_1) the axes remaining parallel to their original directions, equation (6.15) transforms into

$$a(x + x_1)^2 + 2h(x + x_1)(y + y_1) + b(y + y_1)^2 + 2g(x + x_1) + 2f(y + y_1) + c = 0 \tag{6.16}$$

Referred to new axes (6.16) is the equation of two straight lines through origin, and must therefore be a homogeneous quadratic equation in x and y. Simplifying further, we have

$$\begin{aligned} ax^2 + 2hxy + by^2 + 2(ax_1 + hy_1 + g)x + 2(hx_1 + by_1 + f)y + a{x_1}^2 \\ +2hx_1y_1 + b{y_1}^2 + 2gx_1 + 2fy_1 + c = 0 \end{aligned} \tag{6.17}$$

Hence, (6.17) is a homogeneous quadratic equation in x and y with respect to new coordinate axes. The pair of straight lines passes through origin. Hence, the coefficients of x and y and the constant in (6.17) must separately vanish.

Thus

$$ax_1 + hy_1 + g = 0 \tag{6.18}$$

$$hx_1 + by_1 + f = 0 \tag{6.19}$$

and

$$a{x_1}^2 + 2hx_1y_1 + b{y_1}^2 + 2gx_1 + 2fy_1 + c = 0 \tag{6.20}$$

Multiplying (6.18) by x_1, (6.19) by y_1, adding and subtracting the result from (6.20), we get

$$gx_1 + fy_1 + c = 0 \tag{6.21}$$

If we solve (6.18) and (6.19), we find that

$$x_1 = \frac{fh - gb}{h^2 - ab}, \quad y_1 = \frac{fa - gh}{h^2 - ab}$$

If we put these values in (6.21), and simplify, we get

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

This can be expressed in the determinant form as

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

which is the condition that the equation (6.15) should represent a pair of straight lines.

Note: The point of intersection can be found by solving any two of the equations (6.18), (6.19) and (6.21).

We can arrive at the same result by the following alternative method. The equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight lines if the expression on the left can be broken up into two linear factors of the type

$$lx + my + n \qquad \text{and} \qquad l'x + m'y + n'.$$

We then have

$$\begin{aligned} &ax^2 + 2hxy + by^2 + 2gx + 2fy + c \\ &\equiv (lx + my + n)(l'x + m'y + n') \end{aligned}$$

which gives $ll' = a$, $mm' = b$, $nn' = c$, $lm' + ml' = 2h$, $ln' + nl' = 2g$, $mn' + nm' = 2f$.

Multiplying the last three results together, we obtain

$$2ll'mm'nn' + ll'(m^2n'^2 + n^2m'^2) + mm'(n^2l^2 + l^2n'^2) + nn'(l^2m'^2 + m^2l'^2) = 8fgh$$

which reduces to

$$2abc + a(4f^2 - 2ba) + b(4g^2 - 2ac) + c(4h^2 - 2ab) = 8fgh$$

or

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Remark

We have noted in the first proof above that if the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, and if we translate the axes so that the point of intersection is taken as new origin, then this equation reduces to the homogeneous form

$$ax^2 + 2hxy + by^2 = 0$$

in new coordinates. We shall be using this fact quite often in what follows.

Sufficiency of the Condition

We have seen that if the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{6.22}$$

represents two straight lines, then

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \tag{6.23}$$

This shows that (6.23) is the necessary condition that the general equation should represent two straight lines. We shall now show that this condition is sufficient also.

If $\quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ i.e., $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

then this is precisely the condition that the lines

$$\begin{aligned} ax + hy + g &= 0 \\ hx + by + f &= 0 \\ gx + fy + c &= 0 \end{aligned}$$

are concurrent. If these lines are concurrent at the point (x_1, y_1), then the equations (6.18), (6.19) and (6.21) are true. If we multiply (6.18) by x_1, (6.19) by y_1 and add them to (6.21), we get (6.20), so (6.20) is also true. And these indicate that when the origin is transferred to (x_1, y_1), the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

becomes

$$ax^2 + 2hxy + by^2 = 0 \tag{6.24}$$

[See (6.17)]. But we know that (6.24) represents a pair of lines through the (new) origin and so if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines through (x_1, y_1).

6.6 Angle between Two Lines

If the origin is transferred to the point of intersection of the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ the equation reduces to

$$ax^2 + 2hxy + by^2 = 0$$

The lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ are, therefore, parallel to the pair of lines

$$ax^2+2hxy+by^2=0$$

through the origin.

The angle between the given lines is, therefore,

$$\tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$$

The lines are parallel if $h^2=ab$, and perpendicular if $a+b=0$.

EXERCISE 6.3

1. Find what the following equations become when the origin is shifted to the point (1,1)

   (i) $x^2+xy-3x-y+2=0$

   (ii) $xy-y^2-x+y=0$

   (iii) $xy-x-y+1=0$

   (iv) $x^2-y^2-2x+2y=0$

2. Show that the equation $3x^2+7xy+2y^2+5x+5y+2=0$ represents a pair of straight lines.

3. Show that the equation $2x^2-5xy+2y^2-3x+3y+1=0$ represents two straight lines intersecting at an angle θ such that $\tan\theta=\frac{3}{4}$

4. Show that the equation $x^2-y^2-x+3y-2=0$ represents a pair of straight lines. Find them, and show they are at right angles.

5. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, show that they intersect in the point

$$\left(\frac{hf-bg}{ab-h^2},\ \frac{hg-af}{ab-h^2}\right)$$

6. Show that the four lines given by the equations $3x^2+8xy-3y^2=0$ and $3x^2+8xy-3y^2+2x-4y-1=0$ form a square. Find the equations of the diagonals of the square.

TABLES

TABLE I

Four-Place Values of Trigonometric Functions
Angle θ in Degrees and Radians

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
0 00′	.0000	.0000	No value	.0000	No value	1.000	1.0000	1.5708	**90 00′**
10	029	029	343.8	029	343.8	000	000	679	50
20	058	058	171.9	058	171.9	000	000	650	40
30	.0087	.0087	114.6	.0087	114.6	1.000	1.0000	1.5621	30
40	116	116	85.95	116	85.94	000	.9999	592	20
50	145	145	68.76	145	68.75	000	999	563	10
1 00′	.0175	.0175	57.30	.0175	57.29	1.000	.9998	1.5533	**89° 00′**
10	204	204	49.11	204	49.10	000	998	504	50
20	233	233	42.98	233	42.96	000	997	475	40
30	.0262	.0262	38.20	.0262	38.19	1.000	.9997	1.5446	30
40	291	291	34.38	291	34.37	000	996	417	20
50	320	320	31.26	320	31.24	001	995	388	10
2 00′	.0349	.0349	28.65	.0349	28.64	1.001	.9994	1.5359	**88 00′**
10	378	378	26.45	378	26.43	001	993	330	50
20	407	407	24.56	407	24.54	001	992	301	40
30	.0436	.0436	22.93	.0437	22.90	1.001	.9990	1.5272	30
40	465	465	21.49	466	21.47	001	989	243	20
50	495	494	20.23	495	20.21	001	988	213	10
3 00′	.0524	.0523	19.11	.0524	19.08	1.001	.9986	1.5184	**87 00′**
10	553	552	18.10	553	18.07	002	985	155	50
20	582	581	17.20	582	17.17	002	983	126	40
30	.0611	.0610	16.38	.0612	16.35	1.002	.9981	1.5097	30
40	640	640	15.64	641	15.60	002	980	068	20
50	669	669	14.96	670	14.92	002	978	039	10
4 00′	.0698	.0698	14.34	.0699	14.30	1.002	.9976	1.5010	**86 00′**
10	727	727	13.76	729	13.73	003	974	981	50
20	756	756	13.23	758	13.20	003	971	952	40
30	.0785	.0785	12.75	.0787	12.71	1.003	.9969	1.4923	30
40	814	814	12.29	816	12.25	003	967	893	20
50	844	843	11.87	846	11.83	004	964	864	10
5 00′	.0873	.0872	11.47	.0875	11.43	1.004	.9962	1.4835	**85 00′**
10	902	901	11.10	904	11.06	004	959	806	50
20	931	929	10.76	934	10.71	004	957	777	40
30	.0960	.0958	10.43	.0963	10.39	1.005	.9954	1.4748	30
40	989	987	10.13	992	10.08	005	951	719	20
50	.1018	.1016	9.839	.1022	9.788	005	948	690	10
6 00′	.1047	.1045	9.567	.1051	9.514	1.006	.9945	1.4661	**84 00′**
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
6° 00′	.1047	.1045	9.567	.1051	9.514	1.006	.9945	1.4661	84° 00′
10	076	074	9.309	080	9.255	006	942	632	50
20	105	103	9.065	110	9.010	006	939	603	40
30	.1134	.1132	8.834	.1139	8.777	1.006	.9936	1.4573	30
40	164	161	8.614	169	8.556	007	932	544	20
50	193	190	8.405	198	8.345	007	929	515	10
7° 00′	.1222	.1219	8.206	.1228	8.144	1.008	.9925	1.4486	83° 00′
10	251	248	8.016	257	7.953	008	922	457	50
20	280	276	7.834	287	7.770	008	918	428	40
30	.1309	.1305	7.661	.1317	7.596	1.009	.9914	1.4399	30
40	338	334	7.496	346	7.429	009	911	370	20
50	367	363	7.337	376	7.269	009	907	341	10
8° 00′	.1396	.1392	7.185	.1405	7.115	1.010	.9903	1.4312	82° 00′
10	425	421	7.040	435	6.968	010	899	283	50
20	454	449	6.900	465	6.827	011	894	254	40
30	.1484	.1478	6.765	.1495	6.691	1.011	.9890	1.4224	30
40	513	507	6.636	524	6.561	012	886	195	20
50	542	536	6.512	554	6.435	012	881	166	10
9° 00′	.1571	.1564	6.392	.1584	6.314	1.012	.9877	1.4137	81° 00′
10	600	593	277	614	197	013	872	108	50
20	629	622	166	644	084	013	868	079	40
30	.1658	.1650	6.059	.1673	5.976	1.014	.9863	1.4050	30
40	687	679	5.955	703	871	014	858	1.4021	20
50	716	708	855	733	769	015	853	992	10
10° 00′	.1745	.1736	5.759	.1763	5.671	1.015	.9848	1.3963	80° 00′
10	774	765	665	793	576	016	843	934	50
20	804	794	575	823	485	016	838	904	40
30	.1833	.1822	5.487	.1853	5.396	1.017	.9833	1.3875	30
40	862	851	403	883	309	018	827	846	20
50	891	880	320	914	226	018	822	817	10
11° 00′	.1920	.1908	5.241	.1944	5.145	1.019	.9816	1.3788	79° 00′
10	949	937	164	974	066	019	811	759	50
20	978	965	089	.2004	4.989	020	805	730	40
30	.2007	.1994	5.016	.2035	4.915	1.020	.9799	1.3701	30
40	036	.2022	4.945	065	843	021	793	672	20
50	065	051	876	095	773	022	787	643	10
12° 00′	.2094	.2079	4.810	.2126	4.705	1.022	.9781	1.3614	78° 00′
10	123	108	745	156	638	023	775	584	50
20	153	136	682	186	574	024	769	555	40
30	.2182	.2164	4.620	.2217	4.511	1.024	.9763	1.3526	30
40	211	193	560	247	449	025	757	497	20
50	240	221	502	278	390	026	750	468	10
13° 00′	.2269	.2250	4.445	.2309	4.331	1.026	.9744	1.3439	77° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	**Radians**	**sin θ**	**csc θ**	**tan θ**	**cot θ**	**sec θ**	**cos θ**		
13° 00′	.2269	.2250	4.445	.2309	4.331	1.026	.9744	1.3439	**77° 00′**
10	298	278	390	339	275	027	737	410	50
20	327	306	336	370	219	028	730	381	40
30	.2356	.2334	4.284	.2401	4.165	1.028	.9724	1.3352	30
40	385	363	232	432	113	029	717	323	20
50	414	391	182	462	061	030	710	294	10
14° 00′	.2443	.2419	4.134	.2493	4.011	1.031	.9703	1.3265	**76° 00′**
10	473	447	086	524	3.962	031	696	235	50
20	502	476	039	555	914	032	689	206	40
30	.2531	.2504	3.994	.2586	3.867	1.033	.9681	1.3177	30
40	560	532	950	617	821	034	674	148	20
50	589	560	906	648	776	034	667	119	10
15° 00′	.2618	.2588	3.864	.2679	3.732	1.035	.9659	1.3090	**75° 00′**
10	647	616	822	711	689	036	652	061	50
20	676	644	782	742	647	037	644	032	40
30	.2705	.2672	3.742	.2773	3.606	1.038	.9636	1.3003	30
40	734	700	703	805	566	039	628	974	20
50	763	728	665	836	526	039	621	945	10
16° 00′	.2793	.2756	3.628	.2867	3.487	1.040	.9613	1.2915	**74° 00′**
10	822	784	592	899	450	041	605	886	50
20	851	812	556	931	412	042	596	857	40
30	.2880	.2840	3.521	.2962	3.376	1.043	.9588	1.2828	30
40	909	868	487	994	340	044	580	799	20
50	938	896	453	.3026	305	045	572	770	10
17° 00′	.2967	.2924	3.420	.3057	3.271	1.046	.9563	1.2741	**73° 00′**
10	996	952	388	089	237	047	555	712	50
20	.3025	979	357	121	204	048	546	683	40
30	.3054	.3007	3.326	.3153	3.172	1.048	.9537	1.2654	30
40	083	035	295	185	140	049	528	625	20
50	113	062	265	217	108	050	520	595	10
18° 00′	.3142	.3090	3.236	.3249	3.078	1.051	.9511	1.2566	**72° 00′**
10	171	118	207	281	047	052	502	537	50
20	200	145	179	314	018	053	492	508	40
30	.3229	.3173	3.152	.3346	2.989	1.054	.9483	1.2479	30
40	258	201	124	378	960	056	474	450	20
50	287	228	098	411	932	057	465	421	10
19° 00′	.3316	.3256	3.072	.3443	2.904	1.058	.9455	1.2392	**71° 00′**
10	345	283	046	476	877	059	446	363	50
20	374	311	021	508	850	060	436	334	40
30	.3403	.3338	2.996	.3541	2.824	1.061	.9426	1.2305	30
40	432	365	971	574	798	062	417	275	20
50	462	393	947	607	773	063	407	246	10
20° 00′	.3491	.3420	2.924	.3640	2.747	1.064	.9397	1.2217	**70° 00′**
		cos θ	**sec θ**	**cot θ**	**tan θ**	**csc θ**	**sin θ**	**Radians**	**Degrees**
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	**Radians**	**sin θ**	**csc θ**	**tan θ**	**cot θ**	**sec θ**	**cos θ**		
20° 00′	.3491	.3420	2.924	.3640	2.747	1.064	.9397	1.2217	**70° 00′**
10	520	448	901	673	723	065	387	188	50
20	549	475	878	706	699	066	377	159	40
30	.3578	.3502	2.855	.3739	2.675	1.068	.9367	1.2130	30
40	607	529	833	772	651	069	356	101	20
50	636	557	812	805	628	070	346	072	10
21° 00′	.3665	.3584	2.790	.3839	2.605	1.071	.9336	1.2043	**69° 00′**
10	694	611	769	872	583	072	325	1.2014	50
20	723	638	749	906	560	074	315	985	40
30	.3752	.3665	2.729	.3939	2.539	1.075	.9304	1.1956	30
40	782	692	709	973	517	076	293	926	20
50	811	719	689	.4006	496	077	283	897	10
22° 00′	.3840	.3746	2.669	.4040	2.475	1.079	.9272	1.1868	**68° 00′**
10	869	773	650	074	455	080	261	839	50
20	898	800	632	108	434	081	250	810	40
30	.3927	.3827	2.613	.4142	2.414	1.082	.9239	1.1781	30
40	956	854	595	176	394	084	228	752	20
50	985	881	577	210	375	085	216	723	10
23° 00′	.4014	.3907	2.559	.4245	2.356	1.086	.9205	1.1694	**67° 00′**
10	043	934	542	279	337	088	194	665	50
20	072	961	525	314	318	089	182	636	40
30	.4102	.3987	2.508	.4348	2.300	1.090	.9171	1.1606	30
40	131	.4014	491	383	282	092	159	577	20
50	160	041	475	417	264	093	147	548	10
24° 00′	.4189	.4067	2.459	.4452	2.246	1.095	.9135	1.1519	**66° 00′**
10	218	094	443	487	229	096	124	490	50
20	247	120	427	522	211	097	112	461	40
30	.4276	.4147	2.411	.4557	2.194	1.099	.9100	1.1432	30
40	305	173	396	592	177	100	088	403	20
50	334	200	381	628	161	102	075	374	10
25° 00′	.4363	.4226	2.366	.4663	2.145	1.103	.9063	1.1345	**65° 00′**
10	392	253	352	699	128	105	051	316	50
20	422	279	337	734	112	106	038	286	40
30	.4451	.4305	2.323	.4770	2.097	1.108	.9026	1.1257	30
40	480	331	309	806	081	109	013	228	20
50	509	358	295	841	066	111	001	199	10
26° 00′	.4538	.4384	2.281	.4877	2.050	1.113	.8988	1.1170	**64° 00′**
10	567	410	268	913	035	114	975	141	50
20	596	436	254	950	020	116	962	112	40
30	.4625	.4462	2.241	.4986	2.006	1.117	.8949	1.1083	30
40	654	488	228	.5022	1.991	119	936	054	20
50	683	514	215	059	977	121	923	1.1025	10
27° 00′	.4712	.4540	2.203	.5095	1.963	1.122	.8910	1.0996	**63° 00′**
		cos θ	**sec θ**	**cot θ**	**tan θ**	**csc θ**	**sin θ**	**Radians**	**Degrees**
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
27° 00′	.4712	.4540	2.203	.5095	1.963	1.122	.8910	1.0996	63° 00′
10	741	566	190	132	949	124	897	966	50
20	771	592	178	169	935	126	884	937	40
30	.4800	.4617	2.166	.5206	1.921	1.127	.8870	1.0908	30
40	829	643	154	243	907	129	857	879	20
50	858	669	142	280	894	131	843	850	10
28° 00′	.4887	.4695	2.130	.5317	1.881	1.133	.8829	1.0821	62° 00′
10	916	720	118	354	868	134	816	792	50
20	945	746	107	392	855	136	802	763	40
30	.4974	.4772	2.096	.5430	1.842	1.138	.8788	1.0734	30
40	.5003	797	085	467	829	140	774	705	20
50	032	823	074	505	816	142	760	676	10
29° 00′	.5061	.4848	2.063	.5543	1.804	1.143	.8746	1.0647	61° 00′
10	091	874	052	581	792	145	732	617	50
20	120	899	041	619	780	147	718	588	40
30	.5149	.4924	2.031	.5658	1.767	1.149	.8704	1.0559	30
40	178	950	020	696	756	151	689	530	20
50	207	975	010	735	744	153	675	501	10
30° 00′	.5236	.5000	2.000	.5774	1.732	1.155	.8660	1.0472	60° 00′
10	265	025	1.990	812	720	157	646	443	50
20	294	050	980	851	709	159	631	414	40
30	.5323	.5075	1.970	.5890	1.698	1.161	.8616	1.0385	30
40	352	100	961	930	686	163	601	356	20
50	381	125	951	969	675	165	587	327	10
31° 00′	.5411	.5150	1.942	.6009	1.664	1.167	.8572	1.0297	59° 00′
10	440	175	932	048	653	169	557	268	50
20	469	200	923	088	643	171	542	239	40
30	.5498	.5225	1.914	.6128	1.632	1.173	.8526	1.0210	30
40	527	250	905	168	621	175	511	181	20
50	556	275	896	208	611	177	496	152	10
32° 00′	.5585	.5299	1.887	.6249	1.600	1.179	.8480	1.0123	58° 00′
10	614	324	878	289	590	181	465	094	50
20	643	348	870	330	580	184	450	065	40
30	.5672	.5373	1.861	.6371	1.570	1.186	.8434	1.0036	30
40	701	398	853	412	560	188	418	1.0007	20
50	730	422	844	453	550	190	403	977	10
33° 00′	.5760	.5446	1.836	.6494	1.540	1.192	.8387	.9948	57° 00′
10	789	471	828	536	530	195	371	919	50
20	818	495	820	577	520	197	355	890	40
30	.5847	.5519	1.812	.6619	1.511	1.199	.8339	.9861	30
40	876	544	804	661	501	202	323	832	20
50	905	568	796	703	1.492	204	307	803	10
34° 00′	.5934	.5592	1.788	.6745	1.483	1.206	.8290	.9774	56° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	**Radians**	**sin θ**	**csc θ**	**tan θ**	**cot θ**	**sec θ**	**cos θ**		
34° 00′	.5934	.5592	1.788	.6745	1.483	1.206	.8290	.9774	**56° 00′**
10	963	616	781	787	473	209	274	745	50
20	992	640	773	830	464	211	258	716	40
30	.6021	.5664	1.766	.6873	1.455	1.213	.8241	.9687	30
40	050	688	758	916	446	216	225	657	20
50	080	712	751	959	437	218	208	628	10
35° 00′	.6109	.5736	1.743	.7002	1.428	1.221	.8192	.9599	**55° 00′**
10	138	760	736	046	419	223	175	570	50
20	167	783	729	089	411	226	158	541	40
30	.6196	.5807	1.722	.7133	1.402	1.228	.8141	.9512	30
40	225	831	715	177	393	231	124	483	20
50	254	854	708	221	385	233	107	454	10
36° 00′	.6283	.5878	1.701	.7265	1.376	1.236	.8090	.9425	**54° 00′**
10	312	901	695	310	368	239	073	396	50
20	341	925	688	355	360	241	056	367	40
30	.6370	.5948	1.681	.7400	1.351	1.244	.8039	.9338	30
40	400	972	675	445	343	247	021	308	20
50	429	995	668	490	335	249	004	279	10
37° 00′	.6458	.6018	1.662	.7536	1.327	1.252	.7986	.9250	**53° 00′**
10	487	041	655	581	319	255	969	221	50
20	516	065	649	627	311	258	951	192	40
30	.6545	.6088	1.643	.7673	1.303	1.260	.7934	.9163	30
40	574	111	636	720	295	263	916	134	20
50	603	134	630	766	288	266	898	105	10
38° 00′	.6632	.6157	1.624	.7813	1.280	1.269	.7880	.9076	**52° 00′**
10	661	180	618	860	272	272	862	047	50
20	690	202	612	907	265	275	844	.9018	40
30	.6720	.6225	1.606	.7954	1.257	1.278	.7826	.8988	30
40	749	248	601	.8002	250	281	808	959	20
50	778	271	595	050	242	284	790	930	10
39° 00′	.6807	.6293	1.589	.8098	1.235	1.287	.7771	.8901	**51° 00′**
10	836	316	583	146	228	290	753	872	50
20	865	338	578	195	220	293	735	843	40
30	.6894	.6361	1.572	.8243	1.213	1.296	.7716	.8814	30
40	923	383	567	292	206	299	698	785	20
50	952	406	561	342	199	302	679	756	10
40° 00′	.6981	.6428	1.556	.8391	1.192	1.305	.7660	.8727	**50° 00′**
10	.7010	450	550	441	185	309	642	698	50
20	039	472	545	491	178	312	623	668	40
30	.7069	.6494	1.540	.8541	1.171	1.315	.7604	.8639	30
40	098	517	535	591	164	318	585	610	20
50	127	539	529	642	157	322	566	581	10
41° 00′	.7156	.6561	1.524	.8693	1.150	1.325	.7547	.8552	**49° 00′**
		cos θ	**sec θ**	**cot θ**	**tan θ**	**csc θ**	**sin θ**	**Radians**	**Degrees**
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
41° 00′	.7156	.6561	1.524	.8693	1.150	1.325	.7547	.8552	49° 00′
10	185	583	519	744	144	328	528	523	50
20	214	604	514	796	137	332	509	494	40
30	.7243	.6626	1.509	.8847	1.130	1.335	.7490	.8465	30
40	272	648	504	899	124	339	470	436	20
50	301	670	499	952	117	342	451	407	10
42° 00′	.7330	.6691	1.494	.9004	1.111	1.346	.7431	.8378	48° 00′
10	359	713	490	057	104	349	412	348	50
20	389	734	485	110	098	353	392	319	40
30	.7418	.6756	1.480	.9163	1.091	1.356	.7373	.8290	30
40	447	777	476	217	085	360	353	261	20
50	476	799	471	271	079	364	333	232	10
43° 00′	.7505	.6820	1.466	.9325	1.072	1.367	.7314	.8203	47° 00′
10	534	841	462	380	066	371	294	174	50
20	563	862	457	435	060	375	274	145	40
30	.7592	.6884	1.453	.9490	1.054	1.379	.7254	.8116	30
40	621	905	448	545	048	382	234	087	20
50	650	926	444	601	042	386	214	058	10
44° 00′	.7679	.6947	1.440	.9657	1.036	1.390	.7193	.8029	46° 00′
10	709	967	435	713	030	394	173	.7999	50
20	738	988	431	770	024	398	153	970	40
30	.7767	.7009	1.427	.9827	1.018	1.402	.7133	.7941	30
40	796	030	423	884	012	406	112	912	20
50	825	050	418	942	006	410	092	883	10
45° 00′	.7854	.7071	1.414	1.000	1.000	1.414	.7071	.7854	45° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE II

Four-Place Values of Trigonometric Functions
Real Numbers u, or Angles θ, in Radians and Degrees

Real Number u or θ radians	θ degrees	$\sin u$ or $\sin\theta$	$\csc u$ or $\csc\theta$	$\tan u$ or $\tan\theta$	$\cot u$ or $\cot\theta$	$\sec u$ or $\sec\theta$	$\cos u$ or $\cos\theta$
0.00	0° 00′	0.0000	No value	0.0000	No value	1.000	1.000
.01	0° 34′	.0100	100.0	.0100	100.0	1.000	1.000
.02	1° 09′	.0200	50.00	.0200	49.99	1.000	0.9998
.03	1° 43′	.0300	33.34	.0300	33.32	1.000	0.9996
.04	2° 18′	.0400	25.01	.0400	24.99	1.001	0.9992
0.05	2° 52′	0.0500	20.01	0.0500	19.98	1.001	0.9988
.06	3° 26′	.0600	16.68	.0601	16.65	1.002	.9982
.07	4° 01′	.0699	14.30	.0701	14.26	1.002	.9976
.08	4° 35′	.0799	12.51	.0802	12.47	1.003	.9968
.09	5° 09′	.0899	11.13	.0902	11.08	1.004	.9960
0.10	5° 44′	0.0998	10.02	0.1003	9.967	1.005	0.9950
.11	6° 18′	.1098	9.109	.1104	9.054	1.006	.9940
.12	6° 53′	.1197	8.353	.1206	8.293	1.007	.9928
.13	7° 27′	.1296	7.714	.1307	7.649	1.009	.9916
.14	8° 01′	.1395	7.166	.1409	7.096	1.010	.9902
0.15	8° 36′	0.1494	6.692	0.1511	6.617	1.011	0.9888
.16	9° 10′	.1593	6.277	.1614	6.197	1.013	.9872
.17	9° 44′	.1692	5.911	.1717	5.826	1.015	.9856
.18	10° 19′	.1790	5.586	.1820	5.495	1.016	.9838
.19	10° 53′	.1889	5.295	.1923	5.200	1.018	.9820
0.20	11° 28′	0.1987	5.033	0.2027	4.933	1.020	0.9801
.21	12° 02′	.2085	4.797	.2131	4.692	1.022	.9780
.22	12° 36′	.2182	4.582	.2236	4.472	1.025	.9759
.23	13° 11′	.2280	4.386	.2341	4.271	1.027	.9737
.24	13° 45′	.2377	4.207	.2447	4.086	1.030	.9713
0.25	14° 19′	0.2474	4.042	0.2553	3.916	1.032	0.9689
.26	14° 54′	.2571	3.890	.2660	3.759	1.035	.9664
.27	15° 28′	.2667	3.749	.2768	3.613	1.038	.9638
.28	16° 03′	.2764	3.619	.2876	3.478	1.041	.9611
.29	16° 37′	.2860	3.497	.2984	3.351	1.044	.9582
0.30	17° 11′	0.2955	3.384	0.3093	3.233	1.047	0.9553
.31	17° 46′	.3051	3.278	.3203	3.122	1.050	.9523
.32	18° 20′	.3146	3.179	.3314	3.018	1.053	.9492
.33	18° 54′	.3240	3.086	.3425	2.920	1.057	.9460
.34	19° 29′	.3335	2.999	.3537	2.827	1.061	.9428
0.35	20° 03′	0.3429	2.916	0.3650	2.740	1.065	0.9394
Real Number u or θ radians	θ degrees	$\sin u$ or $\sin\theta$	$\csc u$ or $\csc\theta$	$\tan u$ or $\tan\theta$	$\cot u$ or $\cot\theta$	$\sec u$ or $\sec\theta$	$\cos u$ or $\cos\theta$

TABLE II—*continued*

Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ
0.35	20° 03′	0.3429	2.916	0.3650	2.740	1.065	0.9394
.36	20° 38′	.3523	2.839	.3764	2.657	1.068	.9359
.37	21° 12′	.3616	2.765	.3879	2.578	1.073	.9323
.38	21° 46′	.3709	2.696	.3994	2.504	1.077	.9287
.39	22° 21′	.3802	2.630	.4111	2.433	1.081	.9249
0.40	22° 55′	0.3894	2.568	0.4228	2.365	1.086	0.9211
.41	23° 29′	.3986	2.509	.4346	2.301	1.090	.9171
.42	24° 04′	.4078	2.452	.4466	2.239	1.095	.9131
.43	24° 38′	.4169	2.399	.4586	2.180	1.100	.9090
.44	25° 13′	.4259	2.348	.4708	2.124	1.105	.9048
0.45	25° 47′	0.4350	2.299	0.4831	2.070	1.111	0.9004
.46	26° 21′	.4439	2.253	.4954	2.018	1.116	.8961
.47	26° 56′	.4529	2.208	.5080	1.969	1.122	.8916
.48	27° 30′	.4618	2.166	.5206	1.921	1.127	.8870
.49	28° 04′	.4706	2.125	.5334	1.875	1.133	.8823
0.50	28° 39′	0.4794	2.086	0.5463	1.830	1.139	0.8776
.51	29° 13′	.4882	2.048	.5594	1.788	1.146	.8727
.52	29° 48′	.4969	2.013	.5726	1.747	1.152	.8678
.53	30° 22′	.5055	1.978	.5859	1.707	1.159	.8628
.54	30° 56′	.5141	1.945	.5994	1.668	1.166	.8577
0.55	31° 31′	0.5227	1.913	0.6131	1.631	1.173	0.8525
.56	32° 05′	.5312	1.883	.6269	1.595	1.180	.8473
.57	32° 40′	.5396	1.853	.6410	1.560	1.188	.8419
.58	33° 14′	.5480	1.825	.6552	1.526	1.196	.8365
.59	33° 48′	.5564	1.797	.6696	1.494	1.203	.8309
0.60	34° 23′	0.5646	1.771	0.6841	1.462	1.212	0.8253
.61	34° 57′	.5729	1.746	.6989	1.431	1.220	.8196
.62	35° 31′	.5810	1.721	.7139	1.401	1.229	.8139
.63	36° 06′	.5891	1.697	.7291	1.372	1.238	.8080
.64	36° 40′	.5972	1.674	.7445	1.343	1.247	.8021
0.65	37° 15′	0.6052	1.652	0.7602	1.315	1.256	0.7961
.66	37° 49′	.6131	1.631	.7761	1.288	1.266	.7900
.67	38° 23′	.6210	1.610	.7923	1.262	1.276	.7838
.68	38° 58′	.6288	1.590	.8087	1.237	1.286	.7776
.69	39° 32′	.6365	1.571	.8253	1.212	1.297	.7712
0.70	40° 06′	0.6442	1.552	0.8423	1.187	1.307	0.7648
Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ

TABLE II—*continued*

Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ
0.70	40° 06′	0.6442	1.552	0.8423	1.187	1.307	0.7648
.71	40° 41′	.6518	1.534	.8595	1.163	1.319	.7584
.72	41° 15′	.6594	1.517	.8771	1.140	1.330	.7518
.73	41° 50′	.6669	1.500	.8949	1.117	1.342	.7452
.74	42° 24′	.6743	1.483	.9131	1.095	1.354	.7385
0.75	42° 58′	0.6816	1.467	0.9316	1.073	1.367	0.7317
.76	43° 33′	.6889	1.452	.9505	1.052	1.380	.7248
.77	44° 07′	.6961	1.436	.9697	1.031	1.393	.7179
.78	44° 41′	.7033	1.422	.9893	1.011	1.407	.7109
.79	45° 16′	.7104	1.408	1.009	.9908	1.421	.7038
0.80	45° 50′	0.7174	1.394	1.030	0.9712	1.435	0.6967
.81	46° 25′	.7243	1.381	1.050	.9520	1.450	.6895
.82	46° 59′	.7311	1.368	1.072	.9331	1.466	.6822
.83	47° 33′	.7379	1.355	1.093	.9146	1.482	.6749
.84	48° 08′	.7446	1.343	1.116	.8964	1.498	.6675
0.85	48° 42′	0.7513	1.331	1.138	0.8785	1.515	0.6600
.86	49° 16′	.7578	1.320	1.162	.8609	1.533	.6524
.87	49° 51′	.7643	1.308	1.185	.8437	1.551	.6448
.88	50° 25′	.7707	1.297	1.210	.8267	1.569	.6372
.89	51° 00′	.7771	1.287	1.235	.8100	1.589	.6294
0.90	51° 34′	0.7833	1.277	1.260	0.7936	1.609	0.6216
.91	52° 08′	.7895	1.267	1.286	.7774	1.629	.6137
.92	52° 43′	.7956	1.257	1.313	.7615	1.651	.6058
.93	53° 17′	.8016	1.247	1.341	.7458	1.673	.5978
.94	53° 51′	.8076	1.238	1.369	.7303	1.696	.5898
0.95	54° 26′	0.8134	1.229	1.398	0.7151	1.719	0.5817
.96	55° 00′	.8192	1.221	1.428	.7001	1.744	.5735
.97	55° 35′	.8249	1.212	1.459	.6853	1.769	.5653
.98	56° 09′	.8305	1.204	1.491	.6707	1.795	.5570
.99	56° 43′	.8360	1.196	1.524	.6563	1.823	.5487
1.00	57° 18′	0.8415	1.188	1.557	0.6421	1.851	0.5403
1.01	57° 52′	.8468	1.181	1.592	.6281	1.880	.5319
1.02	58° 27′	.8521	1.174	1.628	.6142	1.911	.5234
1.03	59° 01′	.8573	1.166	1.665	.6005	1.942	.5148
1.04	59° 35′	.8624	1.160	1.704	.5870	1.975	.5062
1.05	60° 10′	0.8674	1.153	1.743	0.5736	2.010	0.4976
Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ

TABLE II—*continued*

Real Number u or θ radians	θ degrees	$\sin u$ or $\sin\theta$	$\csc u$ or $\csc\theta$	$\tan u$ or $\tan\theta$	$\cot u$ or $\cot\theta$	$\sec u$ or $\sec\theta$	$\cos u$ or $\cos\theta$
1.05	60° 10′	0.8674	1.153	1.743	0.5736	2.010	0.4976
1.06	60° 44′	8724	1.146	1.784	.5604	2.046	.4889
1.07	61° 18′	.8772	1.140	1.827	.5473	2.083	.4801
1.08	61° 53′	.8820	1.134	1.871	.5344	2.122	.4713
1.09	62° 27′	.8866	1.128	1.917	.5216	2.162	.4625
1.10	63° 02′	0.8912	1.122	1.965	0.5090	2.205	0.4536
1.11	63° 36′	.8957	1.116	2.014	.4964	2.249	.4447
1.12	64° 10′	.9001	1.111	2.066	.4840	2.295	.4357
1.13	64° 45′	.9044	1.106	2.120	.4718	2.344	.4267
1.14	65° 19′	.9086	1.101	2.176	.4596	2.395	.4176
1.15	65° 53′	0.9128	1.096	2.234	0.4475	2.448	0.4085
1.16	66° 28′	.9168	1.091	2.296	.4356	2.504	.3993
1.17	67° 02′	.9208	1.086	2.360	.4237	2.563	.3902
1.18	67° 37′	.9246	1.082	2.247	.4120	2.625	.3809
1.19	68° 11′	.9284	1.077	2.498	.4003	2.691	.3717
1.20	68° 45′	0.9320	1.073	2.572	0.3888	2.760	0.3624
1.21	69° 20′	.9356	1.069	2.650	.3773	2.833	.3530
1.22	69° 54′	.9391	1.065	2.733	.3659	2.910	.3436
1.23	70° 28′	.9425	1.061	2.820	.3546	2.992	.3342
1.24	71° 03′	.9458	1.057	2.912	.3434	3.079	.3248
1.25	71° 37′	0.9490	1.054	3.010	0.3323	3.171	0.3153
1.26	72° 12′	.9521	1.050	3.113	.3212	3.270	.3058
1.27	72° 46′	.9551	1.047	3.224	.3102	3.375	.2963
1.28	73° 20′	.9580	1.044	3.341	.2993	3.488	.2867
1.29	73° 55′	.9608	1.041	3.467	.2884	3.609	.2771
1.30	74° 29′	0.9636	1.038	3.602	0.2776	3.738	0.2675
1.31	75° 03′	.9662	1.035	3.747	.2669	3.878	.2579
1.32	75° 38′	.9687	1.032	3.903	.2562	4.029	.2482
1.33	76° 12′	.9711	1.030	4.072	.2456	4.193	.2385
1.34	76° 47′	.9735	1.027	4.256	.2350	4.372	.2288
1.35	77° 21′	0.9757	1.025	4.455	0.2245	4.566	0.2190
1.36	77° 55′	.9779	1.023	4.673	.2140	4.779	.2092
1.37	78° 30′	.9799	1.021	4.913	.2035	5.014	.1994
1.38	79° 04′	.9819	1.018	5.177	.1931	5.273	.1896
1.39	79° 38′	.9837	1.017	5.471	.1828	5.561	.1798
1.40	80° 13′	0.9854	1.015	5.798	0.1725	5.883	0.1700
Real Number u or θ radians	θ degrees	$\sin u$ or $\sin\theta$	$\csc u$ or $\csc\theta$	$\tan u$ or $\tan\theta$	$\cot u$ or $\cot\theta$	$\sec u$ or $\sec\theta$	$\cos u$ or $\cos\theta$

TABLE II—*continued*

Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ
1.40	80° 13′	0.9854	1.015	5.798	0.1725	5.883	0.1700
1.41	80° 47′	.9871	1.013	6.165	.1622	6.246	.1601
1.42	81° 22′	.9887	1.011	6.581	.1519	6.657	.1502
1.43	81° 56′	.9901	1.010	7.055	.1417	7.126	.1403
1.44	82° 30′	.9915	1.009	7.602	.1315	7.667	.1304
1.45	83° 05′	0.9927	1.007	8.238	0.1214	8.299	0.1205
1.46	83° 39′	.9939	1.006	8.989	.1113	9.044	.1106
1.47	84° 13′	.9949	1.005	9.887	.1011	9.938	.1006
1.48	84° 48′	.9959	1.004	10.98	.0910	11.03	.0907
1.49	85° 22′	.9967	1.003	12.35	.0810	12.39	.0807
1.50	85° 57′	0.9975	1.003	14.10	0.0709	14.14	0.0707
1.51	86° 31′	.9982	1.002	16.43	.0609	16.46	.0608
1.52	87° 05′	.9987	1.001	19.67	.0508	19.69	.0508
1.53	87° 40′	.9992	1.001	24.50	.0408	24.52	.0408
1.54	88° 14′	.9995	1.000	32.46	.0308	32.48	.0308
1.55	88° 49′	0.9998	1.000	48.08	0.0208	48.09	0.0208
1.56	89° 23′	.9999	1.000	92.62	.0108	92.63	.0108
1.57	89° 57′	1.000	1.000	1256	.0008	1256	.0008
Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ

TABLE III

Four-Place Logarithms of Numbers from 1 to 10
To extend the table write the number N as

$N = n \times 10^c$, $1 \leqq n < 10$, c an integer, and use

$\log N = \log n + c.$

n	0	1	2	3	4	5	6	7	8	9
1.0	+0.0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
1.1	.0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
1.2	.0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
1.3	.1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
1.4	.1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
1.5	.1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
1.6	.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
1.7	.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
1.8	.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
1.9	.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
2.0	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
2.1	.3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
2.2	.3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
2.3	.3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
2.4	.3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
2.5	.3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
2.6	.4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
2.7	.4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
2.8	.4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
2.9	.4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
3.1	.4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
3.2	.5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
3.3	.5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
3.4	.5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
3.5	.5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
3.6	.5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
3.7	.5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
3.8	.5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
3.9	.5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
4.0	.6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
4.1	.6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
4.2	.6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
4.3	.6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
4.4	.6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
4.5	.6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
4.6	.6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
4.7	.6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
4.8	.6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
4.9	.6902	6911	6920	6928	6937	6946	6955	6964	6972	6981

TABLE III—*continued*

n	0	1	2	3	4	5	6	7	8	9
5.0	+.6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
5.1	.7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
5.2	.7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
5.3	.7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
5.4	.7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
5.5	.7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
5.6	.7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
5.7	.7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
5.8	.7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
5.9	.7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
6.0	.7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
6.1	.7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
6.2	.7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
6.3	.7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
6.4	.8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
6.5	.8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
6.6	.8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
6.7	.8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
6.8	.8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
6.9	.8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
7.0	.8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
7.1	.8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
7.2	.8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
7.3	.8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
7.4	.8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
7.5	.8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
7.6	.8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
7.7	.8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
7.8	.8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
7.9	.8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
8.0	.9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
8.1	.9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
8.2	.9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
8.3	.9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
8.4	.9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
8.5	.9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
8.6	.9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
8.7	.9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
8.8	.9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
8.9	.9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
9.0	.9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
9.1	.9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
9.2	.9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
9.3	.9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
9.4	.9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
9.5	.9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
9.6	.9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
9.7	.9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
9.8	.9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
9.9	.9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

TABLE IV

Four-Place Logarithms of Trigonometric Functions
Angle θ in Degrees

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
0° 00′	No value	No value	No value	No value	10.0000	10.0000	**90° 00′**
10′	7.4637	12.5363	7.4637	12.5363	.0000	.0000	50′
20′	.7648	.2352	.7648	.2352	.0000	.0000	40′
30′	7.9408	12.0592	7.9409	12.0591	.0000	.0000	30′
40′	8.0658	11.9342	8.0658	11.9342	.0000	.0000	20′
50′	.1627	.8373	.1627	.8373	.0000	10.0000	10′
1° 00′	8.2419	11.7581	8.2419	11.7581	10.0001	9.9999	**89° 00′**
10′	.3088	.6912	.3089	.6911	.0001	.9999	50′
20′	.3668	.6332	.3669	.6331	.0001	.9999	40′
30′	.4179	.5821	.4181	.5819	.0001	.9999	30′
40′	.4637	.5363	.4638	.5362	.0002	.9998	20′
50′	.5050	.4950	.5053	.4947	.0002	.9998	10′
2° 00′	8.5428	11.4572	8.5431	11.4569	10.0003	9.9997	**88° 00′**
10′	.5776	.4224	.5779	.4221	.0003	.9997	50′
20′	.6097	.3903	.6101	.3899	.0004	.9996	40′
30′	.6397	.3603	.6401	.3599	.0004	.9996	30′
40′	.6677	.3323	.6682	.3318	.0005	.9995	20′
50′	.6940	.3060	.6945	.3055	.0005	.9995	10′
3° 00′	8.7188	11.2812	8.7194	11.2806	10.0006	9.9994	**87° 00′**
10′	.7423	.2577	.7429	.2571	.0007	.9993	50′
20′	.7645	.2355	.7652	.2348	.0007	.9993	40′
30′	.7857	.2143	.7865	.2135	.0008	.9992	30′
40′	.8059	.1941	.8067	.1933	.0009	.9991	20′
50′	.8251	.1749	.8261	.1739	.0010	.9990	10′
4° 00′	8.8436	11.1564	8.8446	11.1554	10.0011	9.9989	**86° 00′**
10′	.8613	.1387	.8624	.1376	.0011	.9989	50′
20′	.8783	.1217	.8795	.1205	.0012	.9988	40′
30′	.8946	.1054	.8960	.1040	.0013	.9987	30′
40′	.9104	.0896	.9118	.0882	.0014	.9986	20′
50′	.9256	.0744	.9272	.0728	.0015	.9985	10′
5° 00′	8.9403	11.0597	8.9420	11.0580	10.0017	9.9983	**85° 00′**
10′	.9545	.0455	.9563	.0437	.0018	.9982	50′
20′	.9682	.0318	.9701	.0299	.0019	.9981	40′
30′	.9816	.0184	.9836	.0164	.0020	.9980	30′
40′	8.9945	11.0055	8.9966	11.0034	.0021	.9979	20′
50′	9.0070	10.9930	9.0093	10.9907	.0023	.9977	10′
6° 00′	9.0192	10.9808	9.0216	10.9784	10.0024	9.9976	**84° 00′**
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
6° 00′	9.0192	10.9808	9.0216	10.9784	10.0024	9.9976	**84° 00′**
10′	.0311	.9689	.0336	.9664	.0025	.9975	50′
20′	.0426	.9574	.0453	.9547	.0027	.9973	40′
30′	.0539	.9461	.0567	.9433	.0028	.9972	30′
40′	.0648	.9352	.0678	.9322	.0029	.9971	20′
50′	.0755	.9245	.0786	.9214	.0031	.9969	10′
7° 00′	9.0859	10.9141	9.0891	10.9109	10.0032	9.9968	**83° 00′**
10′	.0961	.9039	.0995	.9005	.0034	.9966	50′
20′	.1060	.8940	.1096	.8904	.0036	.9964	40′
30′	.1157	.8843	.1194	.8806	.0037	.9963	30′
40′	.1252	.8748	.1291	.8709	.0039	.9961	20′
50′	.1345	.8655	.1385	.8615	.0041	.9959	10′
8° 00′	9.1436	10.8564	9.1478	10.8522	10.0042	9.9958	**82° 00′**
10′	.1525	.8475	.1569	.8431	.0044	.9956	50′
20′	.1612	.8388	.1658	.8342	.0046	.9954	40′
30′	.1697	.8303	.1745	.8255	.0048	.9952	30′
40′	.1781	.8219	.1831	.8169	.0050	.9950	20′
50′	.1863	.8137	.1915	.8085	.0052	.9948	10′
9° 00′	9.1943	10.8057	9.1997	10.8003	10.0054	9.9946	**81° 00′**
10′	.2022	.7978	.2078	.7922	.0056	.9944	50′
20′	.2100	.7900	.2158	.7842	.0058	.9942	40′
30′	.2176	.7824	.2236	.7764	.0060	.9940	30′
40′	.2251	.7749	.2313	.7687	.0062	.9938	20′
50′	.2324	.7676	.2389	.7611	.0064	.9936	10′
10° 00′	9.2397	10.7603	9.2463	10.7537	10.0066	9.9934	**80° 00′**
10′	.2468	.7532	.2536	.7464	.0069	.9931	50′
20′	.2538	.7462	.2609	.7391	.0071	.9929	40′
30′	.2606	.7394	.2680	.7320	.0073	.9927	30′
40′	.2674	.7326	.2750	.7250	.0076	.9924	20′
50′	.2740	.7260	.2819	.7181	.0078	.9922	10′
11° 00′	9.2806	10.7194	9.2887	10.7113	10.0081	9.9919	**79° 00′**
10′	.2870	.7130	.2953	.7047	.0083	.9917	50′
20′	.2934	.7066	.3020	.6980	.0086	.9914	40′
30′	.2997	.7003	.3085	.6915	.0088	.9912	30′
40′	.3058	.6942	.3149	.6851	.0091	.9909	20′
50′	.3119	.6881	.3212	.6788	.0093	.9907	10′
12° 00′	9.3179	10.6821	9.3275	10.6725	10.0096	9.9904	**78° 00′**
10′	.3238	.6762	.3336	.6664	.0099	.9901	50′
20′	.3296	.6704	.3397	.6603	.0101	.9899	40′
30′	.3353	.6647	.3458	.6542	.0104	.9896	30′
40′	.3410	.6590	.3517	.6483	.0107	.9893	20′
50′	.3466	.6534	.3576	.6424	.0110	.9890	10′
13° 00′	9.3521	10.6479	9.3634	10.6366	10.0113	9.9887	**77° 00′**
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
13° 00′	9.3521	10.6479	9.3634	10.6366	10.0113	9.9887	**77° 00′**
10′	.3575	.6425	.3691	.6309	.0116	.9884	50′
20′	.3629	.6371	.3748	.6252	.0119	.9881	40′
30′	.3682	.6318	.3804	.6196	.0122	.9878	30′
40′	.3734	.6266	.3859	.6141	.0125	.9875	20′
50′	.3786	.6214	.3914	.6086	.0128	.9872	10′
14° 00′	9.3837	10.6163	9.3968	10.6032	10.0131	9 9869	**76° 00′**
10′	.3887	.6113	.4021	.5979	.0134	.9866	50′
20′	.3937	.6063	.4074	.5926	.0137	.9863	40′
30′	3986	.6014	.4127	.5873	.0141	.9859	30′
40′	.4035	.5965	.4178	.5822	.0144	.9856	20′
50′	.4083	.5917	.4230	.5770	.0147	.9853	10′
15° 00′	9.4130	10.5870	9.4281	10.5719	10.0151	9.9849	**75° 00′**
10′	.4177	.5823	.4331	.5669	.0154	.9846	50′
20′	.4223	.5777	.4381	.5619	.0157	.9843	40′
30′	.4269	.5731	.4430	.5570	.0161	.9839	30′
40′	.4314	.5686	.4479	.5521	.0164	.9836	20′
50′	.4359	.5641	.4527	.5473	.0168	.9832	10′
16° 00′	9.4403	10.5597	9.4575	10.5425	10.0172	9.9828	**74° 00′**
10′	.4447	.5553	.4622	.5378	.0175	.9825	50′
20′	.4491	.5509	.4669	.5331	.0179	.9821	40′
30′	.4533	.5467	.4716	.5284	.0183	.9817	30′
40′	.4576	.5424	.4762	.5238	.0186	.9814	20′
50′	.4618	.5382	.4808	.5192	.0190	.9810	10′
17° 00′	9.4659	10.5341	9.4853	10.5147	10.0194	9.9806	**73° 00′**
10′	.4700	.5300	.4898	.5102	.0198	.9802	50′
20′	.4741	.5259	.4943	.5057	.0202	.9798	40′
30′	.4781	.5219	.4987	.5013	.0206	.9794	30′
40	.4821	.5179	.5031	.4969	.0210	.9790	20′
50′	.4861	.5139	.5075	.4925	.0214	.9786	10′
18° 00′	9.4900	10.5100	9.5118	10.4882	10.0218	9.9782	**72° 00′**
10′	.4939	.5061	.5161	.4839	.0222	.9778	50′
20′	.4977	.5023	.5203	.4797	.0226	.9774	40′
30′	.5015	.4985	.5245	.4755	.0230	.9770	30′
40′	.5052	.4948	.5287	.4713	.0235	.9765	20′
50′	.5090	.4910	.5329	.4671	.0239	.9761	10′
19° 00′	9.5126	10.4874	9.5370	10.4630	10.0243	9.9757	**71° 00′**
10′	.5163	.4837	.5411	.4589	.0248	.9752	50′
20′	.5199	.4801	.5451	.4549	.0252	.9748	40′
30′	.5235	.4765	.5491	.4509	.0257	.9743	30′
40′	.5270	.4730	.5531	.4469	.0261	.9739	20′
50′	.5306	.4694	.5571	.4429	.0266	.9734	10′
20° 00′	9.5341	10.4659	9.5611	10.4389	10.0270	9.9730	**70° 00′**
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
20° 00′	9.5341	10.4659	9.5611	10.4389	10.0270	9.9730	**70° 00′**
10′	.5375	.4625	.5650	.4350	.0275	.9725	50′
20′	.5409	.4591	.5689	.4311	.0279	.9721	40′
30′	.5443	.4557	.5727	.4273	.0284	.9716	30′
40′	.5477	.4523	.5766	.4234	.0289	.9711	20′
50′	.5510	.4490	.5804	.4196	.0294	.9706	10′
21° 00′	9.5543	10.4457	9.5842	10.4158	10.0298	9.9702	**69° 00′**
10′	.5576	.4424	.5879	.4121	.0303	.9797	50′
20′	.5609	.4391	.5917	.4083	.0308	.9692	40′
30′	.5641	.4359	.5954	.4046	.0313	.9687	30′
40′	.5673	.4327	.5991	.4009	.0318	.9682	20′
50′	.5704	.4296	.6028	.3972	.0323	.9677	10′
22° 00′	9.5736	10.4264	9.6064	10.3936	10.0328	9.9672	**68° 00′**
10′	.5767	.4233	.6100	.3900	.0333	.9667	50′
20′	.5798	.4202	.6136	.3864	.0339	.9661	40′
30′	.5828	.4172	.6172	.3828	.0344	.9656	30′
40′	.5859	.4141	.6208	.3792	.0349	.9651	20′
50′	.5889	.4111	.6243	.3757	.0354	.9646	10′
23° 00′	9.5919	10.4081	9.6279	10.3721	10.0360	9.9640	**67° 00′**
10′	.5948	.4052	.6314	.3686	.0365	.9635	50′
20′	.5978	.4022	.6348	.3652	.0371	.9629	40′
30′	.6007	.3993	.6383	.3617	.0376	.9624	30′
40′	.6036	.3964	.6417	.3583	.0382	.9618	20′
50′	.6065	.3935	.6452	.3548	.0387	.9613	10′
24° 00′	9.6093	10.3907	9.6486	10.3514	10.0393	9.9607	**66° 00′**
10′	.6121	.3879	.6520	.3480	.0398	.9602	50′
20′	.6149	.3851	.6553	.3447	.0404	.9596	40′
30′	.6177	.3823	.6587	.3413	.0410	.9590	30′
40′	.6205	.3795	.6620	.3380	.0416	.9584	20′
50′	.6232	.3768	.6654	.3346	.0421	.9579	10′
25° 00′	9.6259	10.3741	9.6687	10.3313	10.0427	9.9573	**65° 00′**
10′	.6286	.3714	.6720	.3280	.0433	.9567	50′
20′	.6313	.3687	.6752	.3248	.0439	.9561	40′
30′	.6340	.3660	.6785	.3215	.0445	.9555	30′
40′	.6366	.3634	.6817	.3183	.0451	.9549	20′
50′	.6392	.3608	.6850	.3150	.0457	.9543	10′
26° 00′	9.6418	10.3582	9.6882	10.3118	10.0463	9.9537	**64° 00′**
10′	.6444	.3556	.6914	.3086	.0470	.9530	50′
20′	.6470	.3530	.6946	.3054	.0476	.9524	40′
30′	.6495	.3505	.6977	.3023	.0482	.9518	30′
40′	.6521	.3479	.7009	.2991	.0488	.9512	20′
50′	.6546	.3454	.7040	.2960	.0495	.9505	10′
27° 00′	9.6570	10.3430	9.7072	10.2928	10.0501	9.9499	**63° 00′**
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
27° 00′	9.6570	10.3430	9.7072	10.2928	10.0501	9.9499	**63° 00′**
10′	.6595	.3405	.7103	.2897	.0508	.9492	50′
20′	.6620	.3380	.7134	.2866	.0514	.9486	40′
30′	.6644	.3356	.7165	.2835	.0521	.9479	30′
40′	.6668	.3332	.7196	.2804	.0527	.9473	20′
50′	.6692	.3308	.7226	.2774	.0534	.9466	10′
28° 00′	9.6716	10.3284	9.7257	10.2743	10.0541	9.9459	**62° 00′**
10′	.6740	.3260	.7287	.2713	.0547	.9453	50′
20′	.6763	.3237	.7317	.2683	.0554	.9446	40′
30′	.6787	.3213	.7348	.2652	.0561	.9439	30′
40′	.6810	.3190	.7378	.2622	.0568	.9432	20′
50′	.6833	.3167	.7408	.2592	.0575	.9425	10′
29° 00′	9.6856	10.3144	9.7438	10.2562	10.0582	9.9418	**61° 00′**
10′	.6878	.3122	.7467	.2533	.0589	.9411	50′
20′	.6901	.3099	.7497	.2503	.0596	.9404	40′
30′	.6923	.3077	.7526	.2474	.0603	.9397	30′
40′	.6946	.3054	.7556	.2444	.0610	.9390	20′
50′	.6968	.3032	.7585	.2415	.0617	.9383	10′
30° 00′	9.6990	10.3010	9.7614	10.2386	10.0625	9.9375	**60° 00′**
10′	.7012	.2988	.7644	.2356	.0632	.9368	50′
20′	.7033	.2967	.7673	.2327	.0639	.9361	40′
30′	.7055	.2945	.7701	.2299	.0647	.9353	30′
40′	.7076	.2924	.7730	.2270	.0654	.9346	20′
50′	.7097	.2903	.7759	.2241	.0662	.9338	10′
31° 00′	9.7118	10.2882	9.7788	10.2212	10.0669	9.9331	**59° 00′**
10′	.7139	.2861	.7816	.2184	.0677	.9323	50′
20′	.7160	.2840	.7845	.2155	.0685	.9315	40′
30′	.7181	.2819	.7873	.2127	.0692	.9308	30′
40′	.7201	.2799	.7902	.2098	.0700	.9300	20′
50′	.7222	.2778	.7930	.2070	.0708	.9292	10′
32° 00′	9.7242	10.2758	9.7958	10.2042	10.0716	9.9284	**58° 00′**
10′	.7262	.2738	.7986	.2014	.0724	.9276	50′
20′	.7282	.2718	.8014	.1986	.0732	.9268	40′
30′	.7302	.2698	.8042	.1958	.0740	.9260	30′
40′	.7322	.2678	.8070	.1930	.0748	.9252	20′
50′	.7342	.2658	.8097	.1903	.0756	.9244	10′
33° 00′	9.7361	10.2639	9.8125	10.1875	10.0764	9.9236	**57° 00′**
10′	.7380	.2620	.8153	.1847	.0772	.9228	50′
20′	.7400	.2600	.8180	.1820	.0781	.9219	40′
30′	.7419	.2581	.8208	.1792	.0789	.9211	30′
40′	.7438	.2562	.8235	.1765	.0797	.9203	20′
50′	.7457	.2543	.8263	.1737	.0806	.9194	10′
34° 00′	9.7476	10.2524	9.8290	10.1710	10.0814	9.9186	**56° 00′**
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach −10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
34° 00′	9.7476	10.2524	9.8290	10.1710	10.0814	9.9186	**56° 00′**
10′	.7494	.2506	.8317	.1683	.0823	.9177	50′
20′	.7513	.2487	.8344	.1656	.0831	.9169	40′
30′	.7531	.2469	.8371	.1629	.0840	.9160	30′
40′	.7550	.2450	.8398	.1602	.0849	.9151	20′
50′	.7568	.2432	.8425	.1575	.0858	.9142	10′
35° 00′	9.7586	10.2414	9.8452	10.1548	10.0866	9.9134	**55° 00′**
10′	.7604	.2396	.8479	.1521	.0875	.9125	50′
20′	.7622	.2378	.8506	.1494	.0884	.9116	40′
30′	.7640	.2360	.8533	.1467	.0893	.9107	30′
40′	.7657	.2343	.8559	.1441	.0902	.9098	20′
50′	.7675	.2325	.8586	.1414	.0911	.9089	10′
36° 00′	9.7692	10.2308	9.8613	10.1387	10.0920	9.9080	**54° 00′**
10′	.7710	.2290	.8639	.1361	.0930	.9070	50′
20′	.7727	.2273	.8666	.1334	.0939	.9061	40′
30′	.7744	.2256	.8692	.1308	.0948	.9052	30′
40′	.7761	.2239	.8718	.1282	.0958	.9042	20′
50′	.7778	.2222	.8745	.1255	.0967	.9033	10′
37° 00′	9.7795	10.2205	9.8771	10.1229	10.0977	9.9023	**53° 00′**
10′	.7811	.2189	.8797	.1203	.0986	.9014	50′
20′	.7828	.2172	.8824	.1176	.0996	.9004	40′
30′	.7844	.2156	.8850	.1150	.1005	.8995	30′
40′	.7861	.2139	.8876	.1124	.1015	.8985	20′
50′	.7877	.2123	.8902	.1098	.1025	.8975	10′
38° 00′	9.7893	10.2107	9.8928	10.1072	10.1035	9.8965	**52° 00′**
10′	.7910	.2090	.8954	.1046	.1045	.8955	50′
20′	.7926	.2074	.8980	.1020	.1055	.8945	40′
30′	.7941	.2059	.9006	.0994	.1065	.8935	30′
40′	.7957	.2043	.9032	.0968	.1075	.8925	20′
50′	.7973	.2027	.9058	.0942	.1085	.8915	10′
39° 00′	9.7989	10.2011	9.9084	10.0916	10.1095	9.8905	**51° 00′**
10′	.8004	.1996	.9110	.0890	.1105	.8895	50′
20′	.8020	.1980	.9135	.0865	.1116	.8884	40′
30′	.8035	.1965	.9161	.0839	.1126	.8874	30′
40′	.8050	.1950	.9187	.0813	.1136	.8864	20′
50′	.8066	.1934	.9212	.0788	.1147	.8853	10′
40° 00′	9.8081	10.1919	9.9238	10.0762	10.1157	9.8843	**50° 00′**
10′	.8096	.1904	.9264	.0736	.1168	.8832	50′
20′	.8111	.1889	.9289	.0711	.1179	.8821	40′
30′	.8125	.1875	.9315	.0685	.1190	.8810	30′
40′	.8140	.1860	.9341	.0659	.1200	.8800	20′
50′	.8155	.1845	.9366	.0634	.1211	.8789	10′
41° 00′	9.8169	10.1831	9.9392	10.0608	10.1222	9.8778	**49° 00′**
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach −10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
41° 00′	9.8169	10.1831	9.9392	10.0608	10.1222	9.8778	**49° 00′**
10′	.8184	.1816	.9417	.0583	.1233	.8767	50′
20′	.8198	.1802	.9443	.0557	.1244	.8756	40′
30′	.8213	.1787	.9468	.0532	.1255	.8745	30′
40′	.8227	.1773	.9494	.0506	.1267	.8733	20′
50′	.8241	.1759	.9519	.0481	.1278	.8722	10′
42° 00′	9.8255	10.1745	9.9544	10.0456	10.1289	9.8711	**48° 00′**
10′	.8269	.1731	.9570	.0430	.1301	.8699	50′
20′	.8283	.1717	.9595	.0405	.1312	.8688	40
30′	.8297	.1703	.9621	.0379	.1324	.8676	30′
40′	.8311	.1689	.9646	.0354	.1335	.8665	20′
50′	.8324	.1676	.9671	.0329	.1347	.8653	10′
43° 00′	9.8338	10.1662	9.9697	10.0303	10.1359	9.8641	**47° 00′**
10′	.8351	.1649	.9722	.0278	.1371	.8629	50′
20′	.8365	.1635	.9747	.0253	.1382	.8618	40′
30′	.8378	.1622	.9772	.0228	.1394	.8606	30′
40′	.8391	.1609	.9798	.0202	.1406	.8594	20′
50′	.8405	.1595	.9823	.0177	.1418	.8582	10′
44° 00′	9.8418	10.1582	9.9848	10.0152	10.1431	9.8569	**46° 00′**
10′	.8431	.1569	.9874	.0126	.1443	.8557	50′
20′	.8444	.1556	.9899	.0101	.1455	.8545	40′
30′	.8457	.1543	.9924	.0076	.1468	.8532	30′
40′	.8469	.1531	.9949	.0051	.1480	.8520	20′
50′	.8482	.1518	9.9975	.0025	.1493	.8507	10′
45° 00′	9.8495	10.1505	10.0000	10.0000	10.1505	9.8495	**45° 00′**
	L cos θ	**L sec θ**	**L cot θ**	**L tan θ**	**L csc θ**	**L sin θ**	**Angle θ**

ANSWERS

EXERCISE 1.1

3. $\{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5),(4,5)\}$.
Inverse relation corresponds to the cartesian product
$\{(4,1),(5,1),(4,2),(5,2),(4,3),(5,3),(5,4)\}$ and corresponds to 'greater than' from B to A.

5. (ii)

7. $f^{-1} = \{(b,a),(d,b),(a,c),(c,d)\}$

8. Commutative and associative

9. Yes.

11. (iii) 2^n

12. (i) $n \to n^2 : N \to N$ is one-to-one but *not* onto.
(ii) $n \to |n| : Z \to N \cup \{0\}$ is onto but *not* one-to-one.
(iii) $n \to |n|^2 : Z \to N \cup \{0\}$ is *neither* one-to-one *nor* onto.

16. $x^4 - 6x^3 + 10x^2 - 3x$

17. (i) $4x^2 - 6x + 1$,
(ii) $2x^2 + 6x - 1$,
(iii) $x^4 + 6x^3 + 14x^2 + 15x + 5$
(iv) $4x - 9$

18. ϕ

19. No. Inclusion is *not* symmetric.

20. 2^{mn}, since $A \times B$ has mn elements and has therefore 2^{mn} subsets.

22. (ii), (iv)

23. (ii)

EXERCISE 2.1

1. (a) $\frac{\pi}{12}$ (b) $\frac{-\pi}{8}$ (c) $\frac{17\pi}{9}$ (d) $\frac{7\pi}{3}$

2. (a) 14°19′ (nearly) (b)-114°33′ (nearly) (c) 420° (d) 150°

3. $\frac{5\pi}{12}$ cm

4. $\frac{20\pi}{3}$ cm

5. 6π

6. 25°12′

7. (a) 11°27′16″ (b) 18°19′38″ (nearly) (c) 29°47′ (nearly)

8. 100°

EXERCISE 2.2

1. $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = -\sqrt{3}$, $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$, $\sec\theta = -2$, $\cot\theta = -\frac{1}{\sqrt{3}}$

2. $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\operatorname{cosec}\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$

3. $\sin\theta = -\frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\operatorname{cosec}\theta = -\frac{5}{3}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = \frac{4}{3}$

EXERCISE 2.3

1. $\frac{220}{221}$, $\frac{171}{221}$, $\frac{220}{21}$

13. $\frac{2}{\sqrt{5}}$, $\frac{1}{\sqrt{5}}$, 2

14. $\sqrt{\frac{2}{3}}$, $\frac{-1}{\sqrt{3}}$, $-\sqrt{2}$

15. $\sqrt{4+\frac{\sqrt{15}}{8}}$, $\sqrt{4-\frac{\sqrt{15}}{8}}$, $4+\sqrt{15}$

16. $\sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{2\sqrt{2}}}$, $\sqrt{\frac{4+\sqrt{2}+\sqrt{6}}{2\sqrt{2}}}$, $-(\sqrt{2}+1)+\sqrt{4+2\sqrt{2}}$

EXERCISE 2.4

1.	(i) .9387	(ii) .7431	(iii) 1.402	(iv) 1.501
2.	(i) 32°30′	(ii) 89°30′	(iii) 88°20′	(iv) 18°20′
3.	(i) .5645	(ii) .4295	(iii) .9037	(iv) .9185
4.	(i) 31°36′	(ii) 87°34′	(iii) 38°45′	(iv) 7°49′

EXERCISE 2.7

1. $\theta = (2n+1)\frac{\pi}{4}$ or $2m\pi$, where $n, m \in I$

2. $\theta = n\pi + (-1)^{n+1}\frac{\pi}{6}$ or $(2m+1)\frac{\pi}{2}$, where $n, m \in I$

3. $\theta = \frac{n\pi}{2}$ or $\frac{m\pi}{2} - \frac{\pi}{8}$ where $n, m \in I$.

4. $\theta = n\pi$ or $2m\pi$, where $n, m \in I$.

5. $\theta = 2n\pi - \frac{\pi}{2}$ or $\frac{2m\pi}{5} - \frac{\pi}{10}$, where $n, m \in I$.

6. $x = 2n\pi + \frac{\pi}{4} \pm A$, where $n \in I$.

7. $\theta = n\pi + (-1)^n \sin^{-1}\left(\frac{-1+\sqrt{17}}{8}\right)$

or $n\pi + (-1)^n \sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right)$, where $n \in 1$.

8. $\theta = \frac{2k\pi}{m+n}$ or $\frac{(2k+1)\pi}{m-n}$, where $k \in I$.

9. $\theta = n\pi - \frac{\pi}{4}$ or $m\pi + \tan^{-1}\frac{1}{2}$, where $n, m \in I$.

10. $\theta = n\pi + (-1)^{n-1}\frac{\pi}{2}$ or $m\pi + (-1)^{n+1}\frac{\pi}{6}$, where $n, m \in I$.

EXERCISE 4.1

1. (i) $\sqrt{97}$ (ii) $2a\left|\sin\left(\frac{\alpha-\beta}{2}\right)\right|$ **4.** 5, −3 **5.** (3,0) **6.** x−y =3

EXERCISE 4.2

1. (a) $\frac{1}{2}$ (b) 18 (c) $\frac{75}{2}$. (d) $\frac{57}{2}$ **3.** x=1 **4.** (7,2) and (1,0) **5.** 5x−4y+1 = 0

EXERCISE 4.3

1. (0,0), (3,−9) **3.** $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ **4.** (−4, −15)
5. 1 : 3 **6.** 1 : 2

EXERCISE 4.4

1. (a) Angle of inclination is acute.
(b) Parallel to the x-axis or coincides with the x-axis.
(c) Angle of inclination is obtuse

2. (a) 0 (b) 1 (c) undefined (d) $\frac{-1}{\sqrt{3}}$

3. (a) 0 (b) -1 (c) $-\frac{1}{6}$

5. (a) Parallel (b) Neither (c) Perpendicular (d) Parallel

6. 1

7. 9

EXERCISE 4.5

1. $5x+3y=9$

2. $x^2-8x-4y+20=0$

3. $y=3x$

4. $y=x+a$ where a is the given distance

5. $\frac{x^2}{5}+\frac{y^2}{9}=1$

6. $y=\frac{2}{3}x$

EXERCISE 5.1

1. $y=2x-5$

2. $y=-2(2x+1)$

3. $5y=3x-15$

4. $y=-2$

5. $y=x\sqrt{3}+2, \quad y=-x\sqrt{3}+2; \quad y=\pm x\sqrt{3}-2; \quad \left(-\frac{2}{3}\sqrt{3},0\right), \quad \left(\frac{2}{3}\sqrt{3},0\right)$

6. $30^\circ, 60^\circ$

7. $3y=2x+12$

8. $3y=5(x+3)$

9. $2x+y=6$ *or* $x+2y=6$

10. $y=4, \quad 2x-3y=0, \quad 2x-y=0$

11. $x=2, \quad 7y=6x+79, \quad 7y=-(6x+65)$

12. $5y=-2x+18$

13. $3y=2x-7$

14. $x - y = 1$

16. (i) $y + x - 3\sqrt{2} = 0$
(ii) $y + \sqrt{3}x - 10 = 0$
(iii) $y - x + 5\sqrt{2} = 0$
(iv) $y = 1$

17. (i) $\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \sqrt{2}$
(ii) $\frac{4}{5}x + \frac{3}{5}y = \frac{9}{5}$
(iii) $x = 4, 4$ (iv) $y = 2, 2$

18. $x - 4y - 7 = 0$

19. $x + y - 5\sqrt{2} = 0$

21. 30°

22. $x = 0$, $\sqrt{3}y - x = 0, (0, -3), \left(\frac{-3\sqrt{3}}{2}, \frac{-3}{2}\right)$

23. Coincident **24.** Intersecting **25.** Parallel **26.** $\frac{16}{5}$ **27.** 2 **28.** 1

EXERCISE 6.1

1. (i) $2x + 29y = 0$
(ii) $13x - 19y = 83$
(iii) $x + 12y = 1$
(iv) $3x - 29y = 29$

2. (i) $42x + 21y = 257$
(ii) $21y = 113$
(iii) $7x = 24$
(iv) $63x + 105y = 781$

3. $15x + 12y = 7$ **4.** $6x - 17y = -24$

EXERCISE 6.2

1. $x = 2y, x = 3y$ **2.** $ax - by = 0$ and $bx + ay = 0$. **3.** $\tan^{-1}\frac{1}{3}$

EXERCISE 6.3

1. (i) $x^2 + xy = 0$
   (ii) $xy - y^2 = 0$
   (iii) $xy = 0$
   (iv) $x^2 - y^2 = 0$

6. $2x = 4y + 1$, $2x + y = 0$

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